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ELEMENTS

OF

ARITHMETIC,

THEORETICAL AND PRACTICAL;

ADAPTED TO THE USE OF SCHOOLS,
AND TO PRIVATE STUDY.

A NEW STEREOTYPE EDITION, REVISED AND CORRECTED.

BY F. Ferdinand R. Rudolph HASSLER, F. A. P. S.

New-York:
PUBLISHED BY G. & C. & H. CARVILL.
1829

SOUTHERN DISTRICT OF NEW-YORK, SS.

L. S. BE IT REMEMBERED, That on the 13th day of May, A. D. 1829, in the 53rd year of the Independence of the United States of America, F. R. HASSLER, of the said District, hath deposited in this office the title of a Book, the right whereof he claims as Author, in the words following, to wit:

Elements of Arithmetic, Theoretical and Practical; adapted to the use of Schools, and to Private Study. A New Stereotype Edition, revised and corrected. By F. R. HASSLER, F. A. P. S.

In conformity to the Act of Congress of the United States, entitled "An Act for the encouragement of Learning, by securing the copies of Maps, Charts, and Books, to the authors and proprietors of such copies, during the time therein mentioned." And also to an Act, entitled "An Act, supplementary to an Act, entitled an Act for the encouragement of Learning, by securing the copies of Maps, Charts, and Books, to the authors and proprietors of such copies, during the times therein mentioned, and extending the benefits thereof to the arts of designing, engraving, and etching historical and other prints."

FRED. J. BETTS,
Clerk of the Southern District of New-York.

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INTRODUCTION.

Arithmetic contains the first elements of reasoning upon *Quantity*; its principles take their rise in ideas so simple as to be adapted to the most untutored mind, and to the lowest capacity. It is at the same time so indispensable for every human being, not only in common life, but in the pursuits of the highest sciences, that it forms the most proper, and has always formed one of the principal, branches of the earlier education of youth.

By its very nature it furnishes the means of developing the reasoning faculties, from the time of their first beginning to expand, and of habituating them to correctness and precision. It therefore gives the human mind the power and disposition to reason upon *sound* and *correct Principles*.

It is therefore the duty of the faithful teacher of youth, (not the mere teacher for his own private emolument,) to take advantage of this property of arithmetic, and apply it to cultivate the mind, and enlighten the understanding of his scholars, by a proper reasoning in this elementary science; he should not make it the object of the memory alone; a method that leaves no impression upon the mind, whose results are in consequence lost again as soon as the school is dismissed.

O/10-12-37 Miller

To neglect to take this advantage of the study of arithmetic, is either a proof of ignorance, or an actual dereliction of duty. This may appear strong to many people, but strength is the essential property of truth. I can safely appeal to those who have in early youth been taught by the negligent method of mere rules, and have at a later period attained scientific eminence, to decide between this and any contrary assertion.

The difficulties that the young experience on entering upon any scientific studies, in colleges, or otherwise, are well known; the path to be followed there, *must be that of reasoning*, and no preparations are made for this by their previous education; for the cultivation of the memory alone, is, from the very constitution of the human mind, always detrimental to the reasoning faculty.

However, the opportunity, as has been stated, exists, of cultivating the reasoning faculty at an earlier period, by familiarizing the scholar with the simple reasonings of elementary arithmetic. The step from that to higher or general arithmetic, usually called *Algebra*, becomes by this mode, both short and simple, as in its nature it really is; and the scholar who does not wish to go farther than common arithmetic, can alone obtain the knowledge of the propriety or principles of its application to any occurrence in common life, by a knowledge of it, founded upon correct reasoning. It is entirely wrong to say, and act upon the ground, "*I want to know how to do* this or that," the principle must be, "*I wish to understand* this or that," if ever any lasting good result shall be obtained.

My object in undertaking this work, may be stated as follows:

1st. I wish to smooth the path of the teacher and the scholar, by explaining and proving, the propriety and correctness of any step that is taken, by previous reasonings, leading to the discovery of the principle that ought to direct it, and therefore pointing out the rule for the appropriate operation; and I have, therefore, not been content to give the final result or *Rule*, alone, and the example for its proof, which is an individual, and consequently a defective proof, while reasoning always leads to general propositions and proofs. In this way we attain, step by step, to the real scientific structure of this elementary science, and thus all the operations become satisfactory to the mind, and therefore agreeable to the growing intellect of the scholar.

It is a mistake to suppose that children, at the age at which it is proper to introduce them into this study, do not possess sufficient reasoning faculty to follow a regular *systematic course* of ideas; they will do that with much greater facility, than commit to memory, and duly apply, *detached Rules*, and *unconnected Ideas.* One idea producing the other by its connection with it, there exists a mutual support and assistance between them, which detached rules have not. This underrating of the faculties of youth, is an unhappy result of the habits of older persons, who having been led through many useless difficulties in the progress of their own studies, consider them as indispensable, because they have become habitual to themselves.

In carrying such a system through the whole extent, to that point where more general and extensive considerations, of a higher analytic nature, are to guide us, I have even thought it possible to make a treatise, which a man of science might look at with some satisfaction, and by which the young scholar would arrive at the entrance of his higher scientific studies, properly prepared by a correct habit of reasoning.

2nd. The young and untutored mind, in truth, reasons analytically; a boy, and in fact a man, asks always ***WHY?*** and as he enters more and more deeply into the investigation, continues to ask the reason of every thing that is said to him, in the way of explanation. This is owing to the nature of his situation; he cannot proceed synthetically, for synthesis needs some previous data, averred or adopted, on which to build the reasoning to arrive at a conclusion; this does not yet exist at this early stage of instruction.

In following this mode, and grounding every conclusion upon inquiry, I intend to make a book which a lad, remote from cities, although he might not have had the benefit of a good early education, can take in hand usefully; and which merely a knowledge of reading, coupled with his own desire for instruction, would induce him to undertake as a study, both useful and agreeable; useful, because it would show him the means of accounting to himself for the result of his own labours; and agreeable, because it would afford him a pleasing occupation for his winter evenings. I should be delighted to see several such lads, passing an evening together, with this book among them, each his slate and pencil before him, discussing, mutually giving and solving, the questions which they learn from it, to form out of the occurrences around them. I can promise them more satisfaction from it, than passing that time in the barroom of a public house; and more beneficial, economical results, from the expenditure in book, slate, and pencil, to assist their studies: (for they must write every thing,) than were they to lay out the cost in the vile liquor, that emptiness of mind leads them to call for; they will soon be able to calculate: that they even make a saving, if they write their full studies, ideas, and questions, on paper, with pen and ink, in comparison with the expenses of the deleterious pleasures of a bar-room. If I should succeed only in this part of my aim, I would consider my labour as sufficiently rewarded; and I would have the greatest enjoyment, to meet with such a company, afford them assistance, and partake of their rational amusement.

For the use of this book, I should like to advise, the teacher, as well as the student; first to peruse attentively the theoretical principles of any rule or subject, and then exercise his scholars, or himself, in the application, which will give him an opportunity to generalize, and clear up their, or his, ideas properly; and after having gone through any of the principal subdivisions, to take a general view of the whole of it; taking care to comprehend the leading principles, and the mode of considering the subject that has been treated of; in this way he will be enabled to make a proper use of it in the parts to be treated next.

It is an unavoidable condition in every systematic work, that the subsequent parts shall be grounded upon the preceding ones, and consequently these must be supposed known in the progress of the work, as it proceeds. Therefore also the study of no systematic and good work, can be begun in any other part than at the beginning, by any scholar: that is, a person not fully acquainted with the whole subject of the book, but seeking instruction from it. If any person thinks he already knows some of the elementary parts, and wishes to study only the subsequent part, it is necessary for him to read over, attentively, the parts with which he is acquainted, to make himself acquainted with the manner in which the author expresses himself upon those subjects, upon which he has his own ideas. By comparing these together, he will afterwards be able to understand those parts with which he is not acquainted; and therefore read and study with success; which otherwise will certainly not be the case. This is no more than is *necessary* in every species of intercourse between men, namely, that they be acquainted with each others' language.

F. R. HASSLER.

New-York, October, 1826.

PART I.

FIRST ELEMENTS AND DEDUCTION OF THE FOUR RULES OF ARITHMETIC.

CHAPTER I.

Fundamental Ideas of Quantity.—System of Numeration.

§ 1. QUANTITY, which is the object of Arithmetic, is the *Idea* that has reference to any thing whatever, arising from the consideration of its being susceptible of being more or less; without regard to the *nature* or *kind* of the thing itself. It is not, therefore, an absolute existence nor a quality; but a relative idea, that can be referred to any object whatever.*

§ 2. No quantity, therefore, can be called great or small, much or little, in itself; it can be so only in relation to another quantity of the same kind, which might be greater or less than that quantity, and to which it would be compared.

§ 3. Objects of different kinds cannot be compared with each other directly by their quantity only. When, therefore, Objects of different kinds are to be considered in Arithmetic, it becomes necessary: that a certain relation be given between them, which is completely arbitrary as to quantity itself, and must be determined before any comparison can take place.*

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* So for instance, comparing 3 things or objects to 7 things or objects; be these things or objects, houses, horses, apples, or whatever they be, arithmetic considers only the numbers, and the result presented will, in any case, refer to the objects they were originally to represent.

§ 4. The mutual relation of quantities of things or objects to each other, under certain given conditions, is the object of arithmetic. In this general acceptation then it admits any number of systems of combination, that the imagination can devise.

§ 5. To obtain a clear and distinct idea of arithmetic it is necessary, to impress the mind fully with these fundamental ideas, and the general principles that follow from them, by comparing every operation of arithmetic with them; they will thus become at every step more and more clear and useful; the whole system of arithmetic will become the more simple, the more its principles are generalized.

§ 6. Common arithmetic, which might also be called with propriety, determinate arithmetic, is limited to the most simple combinations of quantities and these are all grounded successively upon the first elementary idea of increase or decrease, or more or less, either simple or repeated successively, or according to a certain determined method, or law.

§ 7. To express quantities we make use, in our system of common arithmetic, of ten figures only, by the means of which, and by their relative places, according to a certain law, we can express any quantity whatsoever. This law is called the system of numeration; and in particular the decimal system, from the circumstance, of its using ten different figures, nine of which are significant, and the tenth indicates the absence of the quantity (or thing, or object.)

§ 8 These figures are in regular succession 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0; this last is used to denote the absence of a quantity; the 1, denotes the unit of any object, in arithmetic called *one*, of whatever kind or nature it may be; the subsequent figures denote in regular succession, each one object more than the one before it.

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* So can, for instance, houses and horses not be compared together, unless a value be assigned to them, for instance, in money; and this at once explains the idea of money as a means of representing every object susceptible to become property, and thence needed to be compared with others not of the same kind.

§ 9. To denote quantities which exceed the number of significant figures, (or above 9,) recourse is had to a law that assigns superior values to those figures, according to the order in which they are placed, assigning to them a value as many times greater, in every successive change of place from the right to the left, as the number of figures indicates; and therefore in our usual system of ten figures, a tenfold value. This must necessarily be the law if the system be able to express all quantities, because any other law, giving another relation of value to the places than the number of figures, would either leave a space of quantity unexpressed, or occasion double expressions, if it were to increase in a greater or less ratio than that number. (The circumstance of this increase taking place from the right towards the left originates in the fact, that this system is borrowed from the Arabic, or rather Asiatic, nations, who have the habit of writing from the right towards the left, instead of our writing from the left towards the right.)

§ 10. Thence we have for the successive values of the numbers, in their successive places from the right to the left, the denominations shown in the following table:

Tens of Billions.	Billions.	Hundred Thousand Millions.	Ten Thousands of Millions.	Thousands of Millions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.
1	1,	1	1	1,	1	1	1,	1	1	1,	1	1	1

Such would be the value, or denomination, of any figure, placed in any one of the places; and if no quantity of one or the other of these denominations is to be expressed, the place of it must be supplied with a 0, in order to give to the next figure its proper rank, and thereby value.

§ 11. In reading the numbers we follow our usual way of reading, and therefore express the greatest quantities first; to render this reading more easy it is also customary in large numbers: to divide off by a (,) every three figures from the right to the left, which divides them by hundreds; so for example:

689,347 would read thus:

Six hundred and eighty-nine *thousands*, three hundred and forty-seven, (understood *units*.)

13,842,167 reads thus:

Thirteen *millions*, eight hundred and forty-two *thousands*, one hundred and sixty-seven.

Read the following numbers:

1st.	73,064;	9th.	94,070,790
2d.	101,070,101;	10th.	4,399,080,502
3d.	500,007;	11th.	100,010,007
4th.	90,807,060,501;	12th.	7,070,409
5th.	1,897,510,234;	13th.	1,902,010,571
6th.	3,904,621;	14th.	70,070,432
7th.	70,009;	15th.	985,007
8th.	501,030;	16th.	101,074,000

§ 12. It may be easily conceived, that other systems might be formed upon the same principles, and of course with the same properties, as to the expression of the greater quantities by the successive rank or place of the figures, and with any other greater or smaller number of significant figures; besides the (0,) which must, like the unit, make part of every such system of numeration.

If no other figures were used but (1) and (0,) that is presence or absence of the quantity indicated by any rank or place of figure, the value in each place will always be successively *double* of that in the preceding place, and the whole of the calculation would become a mechanical change of places;* so for instance, in this system the following numbers 111101, transcribed into our usual decimal system, would be 32, 16, 8, 4, and 1; or (61.) It will be a very good exercise for the reflection of the scholar to try some of this kind of expressions in different systems.

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* This system is called the *Diadik*, invented by Leibnitz, expressly to elucidate particularly the effect of the system in Numeration, as in it all effect of proper value of the numbers is taken away,

§ 13. It may also assist in rendering the principles of the decimal system more clear, to contrast it with the old Roman system of numeration. This consists in the use of seven letters having each a particular signification, as:

M, for one thousand
D, — five hundred
C, — one hundred
L, — fifty
X, — ten
V, — five
I, — unity.

In this system, therefore, the numeration consists merely in writing as many of these letters as will make out the quantity desired; and the whole arithmetic consisted in placing or taking away, upon a black board, as many marks under the denomination of each of these letters, as the calculation required. A bad habit of the Romans, in later ages, introduced into this system anomalies, arising from their considering one of the figures of an inferior number, when placed before a higher one, as subtracted, or taken away, from it, as for example; IV was written for four, XC for ninety, and so on.*

CHAPTER II.

General Ideas, and Notations of [illegible] Arithmetic.

§ 14. The first and simplest combination of qua
and therefore, also the first and simplest operat

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and any value whatever is expressed by denoting the presence or absence of any rank of quantity by the 1, or 0, placed in the rank, to express which, our language has no words, because this system is not in common use.

* Some examples of this will appear in the book of questions, as it would be too long to introduce them here.

arithmetic, from which all others proceed, is called *addition.* Of this we have already an example of the simplest kind in the system of figures, that presents the successive additions of unity in their regular order of succession, and therefore also presents the combination of the quantities by addition as far as the sum 9. *Addition consists* therefore *in finding a quantity*, or number, *equal to two or more other quantities taken together;* or, as it is usually called: to find the *sum* of two or more numbers.

§ 15. If on the contrary *the difference of two quantities* or numbers is *to be found*, the operation *is called subtraction.* In this operation the smaller of two numbers, which is called the *subtrahend*, is taken away from the larger one, and the result is called the *remainder;* it is evidently the opposite of the foregoing. Only two quantities or numbers can be concerned in a subtraction, for one result; if more numbers are to be subtracted, it must be done by a new operation.

§ 16. All the subsequent operations in arithmetic are combinations of the two preceding ones according to certain laws.

§ 17. *The addition of the same number or quantity a certain number of times, is called multiplication.* When this is treated in detail, the manner in which the principles of multiplication are deduced from those of addition, will be shown. The two numbers multiplied into each other are called *factors*, and the result is called the *product.*

§18. *The opposite of the operation of multiplication, is called division.* It represents a successive subtraction of the same number, a certain number of times, from another. The number from which this successive subtraction is made, is called the *dividend;* the number repeatedly subtracted, the *divisor;* and the result the *quotient;* it indicates how many times the divisor is contained in, or can be taken away from, the dividend.

§ 19. These four operations of arithmetic, *Addition, Subtraction, Multiplication,* and *Division, are* called *the four Rules of arithmetic.* It has been observed, that the second is the opposite of the first, and the fourth the opposite of the third; and such must be the case in any system of combination of quantity that can be devised. In all arithmetic, it is always necessary, that both the direct and inverse operation shall be devised; and directions or rules deduced and given for their execution.

§ 20. To facilitate the expression of the idea of these four operations, or rules of arithmetic, certain signs are made use of, to indicate them in an abridged manner, which it is proper and very useful to understand at the very beginning; their use will conduce to clearness in the expression of the operations of arithmetic.

To denote an addition, the sign (+) is used, as for instance: if 7 and 2 are to be added, this will be written, 7+2, and in the same way for more numbers.

To denote a subtraction, the sign (—) is used, so for instance: to indicate that from the number 7, the number 2 is to be subtracted, this will be written 7—2.

To indicate a multiplication, the numbers are separated by a full stop, (.) or by this sign, (×,) thus to indicate the multiplication of 7 by 4, we write 7.4 or 7×4. If two or more quantities already united by + or — are to be affected by the multiplication with one number, these quantities are inclosed in, ( ) and the multiplier written to them, in the same manner as before to the single number, for instance (7+5) × 13 is the sum of 7 and 5 to be multiplied by 13.

To indicate a division, two different signs are also made use of; either by placing two dots, or the colon, (:) after the dividend, and writing the divisor after; or by writing the divisor under the dividend, separating them by a horizontal line, thus 8:2 or $\frac{8}{2}$ denotes that 8 is to be divided by 2.

Besides these four signs we are yet in need of a sign to express the equality of two quantities; this is done (by two horizontal parallel lines) thus, =.

These signs will suffice here; for other forms of calculation, or combination, other signs are made use of; but it will be much easier to understand their meaning when the subject itself is treated; it is therefore more proper to postpone their explanation for the present.

§ 21. As it will be proper for the scholar to exercise himself in the expression of these signs, in order that he may become familiar with their import, and acquire clear ideas of arithmetical operations, I shall here join a few examples of the four rules of arithmetic, to which the teacher may add more.

In Addition. $7+9=16$, means the addition of seven and nine is equal to 16, or the sum of 7 and 9, is 16. $7+3+8=18$; or the addition of seven, and three, and eight, is equal to 18, or the sum of 7, and 3, and 8, is 18.

In Subtraction. $13-7=6$, means the difference between 13 and 7, is equal to 6; or 7 taken from 13 leaves 6; for example: we shall have, joining both the preceding notations, $13+9-8=22-8=14$; which, as is shown by the above example, it will be easy for any scholar to express in words. But the idea conceived as it is written by signs is the best mode of expressing it.

In Multiplication. Expressed first as a repeated addition, then by the appropriate sign between the two factors, and lastly its result, or the Product, to which it is equal is presented in the following example :

$$7+7+7+7=4\,.\,7=4\times 7=28.$$
$$5+5+5+5+5=5\,.\,5=5\times 5=25.$$

In Division. If we express division by a successive subtraction of one number a certain number of times from another, we shall, in the case of this subtraction exhausting the number, reduce it to 0, and thereby show that the divisor is contained in the dividend, exactly as many times as it has been possible to subtract it, so we would have for instance,

$$30-6-6-6-6-6=0.$$

Which showing that six subtracted five times from 30, and its successive remainders leaves nothing; therefore, if we express this as a division, having the result, or quotient, as it is called, on the other side of the sign of equality, we obtain in this case the expression

$$30 : 6 = \tfrac{30}{6} = 5$$

If the successive subtraction of the divisor, should at last leave a number smaller than this divisor, it will give what is called a remainder, that is still affected with the sign of division by the divisor; as for instance in the following example:

$$36 - 8 - 8 - 8 - 8 = \tfrac{36}{8} = 4 + \tfrac{4}{8}$$

This last part of the expression indicates a division that can no longer be executed, on account of the divisor being greater than the dividend; it no longer gives a whole quantity in the result; these expressions are called fractions, and we thus have already the fundamental idea of a fraction, from which we shall hereafter deduce the principles of calculation that are adapted to them.

§ 22. By means of these explanations of the principles, and the notations of arithmetic, it is proper for the teacher to introduce his scholars to the subject, and prepare them for its future practical application, if he would not make it a study toilsome to the boy, and an equally toilsome task for himself. No teacher ever had a scholar who did not ask him (*why?*) when he directed him to do something; and this why, the reasonable and faithful teacher must answer in a satisfactory manner; this will be rendered easy by the preceding process, elucidating the principles of arithmetic. The reasoning of the child must be cultivated, if he is ever *actually* to understand arithmetic, and not forget it when out of school, or out of practice, as will be the case, if he has only committed to memory dead rules for which he saw no reason. By such a process arithmetic will ever be agreeable to the scholar, as an exercise of his intellect within the limits of his capacity. The time spent in explaining and reasoning with the scholar upon these principles will be amply gained by his more successful and regular progress in arithmetic, when applying it to each individual rule and case.

CHAPTER III.

The four rules of Arithmetic in whole numbers.

§ 23. ADDITION has been defined as the method of finding a quantity equal to two or more quantities taken together. Its expression as a problem, is, therefore : *to find* the sum of two or more quantities. From what has been said of the principles of the system of numeration, in common arithmetic, it follows, that in order to prepare the given numbers for addition, they must be written under each other, so as to bring the units of the one under the units of the other ; and so all the numbers successively higher in the order of the system of numeration, will each come under its equal denomination ; by which means they may be added the more easily.

Then the numbers are added together, in this order, beginning always with the unit, and proceeding until we reach the last on the left hand side.

Example.—To find the value of 176873 + 34719.

Write these numbers thus ; 176873
34719
and draw a line beneath them ; ———
then add the column of units, 12
tens, 8
hundreds, 15
thousands, 10
ten thousands, 10
hundreds of thousands, 1
———
211592

placing each particular sum so that the figure on the right hand shall be under the numbers added ; then draw a line and add the numbers as they are now placed. The result thus obtained will be the sum of the numbers added. It is evident here, that whenever the sum of any one of these individual additions exceeds what can, in our system of notation, be written with a single figure, we had to place the figure coming to the left of it, under the next higher order ; and in the second addition, these numbers were then added to the result of the addition next following. This can therefore be done at once, by the following process.

Having, as in the example, found the first sum, 9 and 3, which is 12, (or $9 + 3 = 12$) the 2 is placed under the unit, and the 1 is kept in memory, to be added to the next operation, in this case to the sum of the tens, (this operation is called carrying;) so that in this next addition you say: $7 + 1 + 1 = 9$ or 7 and 1 is 8, (as marked in the example,) and 1 carried gives 9, which is immediately written to the left of the former result, or under the tens; this number can be written entirely, and therefore gives nothing to carry. The next or hundreds would give $8 + 7 = 15$, or 8 and 7 is 15; write 5 and keep 1; then the next, 6 and 4 is 10, and 1 kept is 11; (or $6 + 4 + 1 = 11$;) and so on to the last figure on the left hand.

§ 24. If a greater number of figures is to be added, the same mode of operation is used, only repeated as often as the number of figures given will require; as for instance in the following example:

To find the sum of $67421 + 389 + 641827 + 30 + 4 + 7259 =$

Write these numbers all under each other so that the units fall in the same column, and the other numbers successively under their respective places, thus:

$$\begin{array}{r} 67421 \\ 389 \\ 641827 \\ 30 \\ 4 \\ 7259 \\ \hline 716930 \end{array}$$

Then, having drawn a line beneath, begin again by saying, in the column of the units, $9 + 4 + 0 + 7 + 9 + 1 = 30$, write 0, and keep 3; then for the second column, or that of the tens, say: $3 + 5 + 3 + 2 + 8 + 2 = 23$; write three and keep two, and proceeding in this manner to the last figure on the left hand; which will produce the sum found in the example, under the line. It is necessary to practice such examples sufficiently, until the scholar can execute them with facility and accuracy, so that it becomes to him an easy mechanical practice. It is proper to mix the numbers of different orders, as above, at once, and not to distinguish separate cases, in order that the scholar may seize the principles of the operation intellectually, and with reflection, and not by mere memory and habit.

Examples.—Give the value of the following numbers:

1. 1,006,052 + 70,401 + 3,040,107 + 9,080,071,402 =
2. 17,040,109 + 50,201 + 701 + 30 + 5,000,127 =
3. 70904 + 398125 + 8079123 + 98162753 =
4. 37 + 90005 + 1009645 + 309047 =
5. 773 + 104462 + 34983 + 81090406 =
6. 10,506 + 772 + 15,000,101 + 6,052 =
7. 52 + 79 + 3,031,001 + 7,679 + 5,839 =
8. 304 + 9,192 + 7,000,000 + 5,010,609 =
9. 909 + 9,999 + 9,898,909 + 98,648 =

§ 25. SUBTRACTION, as has been already said, is the opposite of addition; its Problem is: to find the difference between two numbers.

In common arithmetic it is always required, that the number to be subtracted be smaller than the number from which it is to be subtracted; otherwise the result would become, what in universal arithmetic is called negative; that is to say: in denying the possibility of the subtraction, it would indicate the number from which it was intended to be subtracted, to be so much too small to admit this subtraction, as the number found, indicates.

This operation is necessarily limited to two numbers, or quantities; if more should be concerned in a question, the result must be obtained by a repetition of the operation.

§ 26. Of this operation in simple numbers we have given the principle in the explanation of the signs, as in the case of addition; when the numbers are larger the following is the preparation and the operation.

Write the number from which the subtraction is to be made first, and the subtrahend under it, in such a manner that the unit comes under the unit, and the following numbers, to the left, each under its similar superior number, and draw a line under them, thus;

$$\begin{array}{l} 9643187 \\ 7532043 = \text{Subtrahend,} \\ \hline 2111144 = \text{Remainder,} \\ \hline 9643187 = \text{Proof.} \end{array}$$

then take the difference between each of the corresponding numbers, beginning by the unit, and write the difference directly under these numbers; the number resulting therefrom will be the entire difference between the two given numbers.

As well from the principle that this operation is the opposite of the addition, as from the consideration of the preceding operation, it may easily be observed: that the proof of the correct execution of this operation may be given, by *adding* the result, or remainder obtained, to the lower number above the line, or the subtrahend; which addition must give the first or upper number for its result. It is therefore proper to accustom beginners to make this proof, in order that they may have the satisfaction of verifying the correctness of their operation; drawing therefore a second line under the result, the two numbers immediately above are added; when the first number must again appear in the result.

§ 27. In this operation it may evidently occur: that, though the quantity from which another is to be subtracted may be greater, some of the individual numbers, of the inferior order, in the subtrahend, may be larger than those corresponding to them in the superior number.

In this case it becomes necessary to supply the want by borrowing an unit from the next higher order of the upper number, which will then represent a ten in its corresponding order next inferior in place and value, and furnishing of course always in addition to this number itself a larger number than that in the subtrahend, will admit the latter to be taken from it; the remainder is then written in its proper place; and if even the preceding superior number were an 0, the lending being considered as possible from the preceding higher order, the operation would be the same, an unit would be borrowed from it, and the number afterwards called 9 ; again under the supposition before made of the lending being made from the next higher order, which, when reached, is considered as diminished by an unit. It is evident that if the superior number is larger than the inferior or subtrahend, this lending will always be compensated before the end of the operation, whatever be its extent, through the figures preceding the last on the left hand side.

Let the following example be given.

600198056 — 356499278

Place the example as indicated, thus :

$$\begin{array}{r}
\cdot\,\cdot\,\cdot\,\cdot\,\cdot\,\cdot\,\cdot\,\cdot \\
600198056 \\
356499278 \\
\hline
243698778 \\
\hline
600198056
\end{array}$$

Here in the units the 8 cannot be taken from the 6, an unit is therefore borrowed from the 5, in the tens preceding the 6, which added to the 6 gives 16, from which the 8 being taken, leaves 8, to be written in the place of the units. (For beginners it will be proper to mark every figure from which an unit has thus been borrowed, by a dot above it, which is done in order that it may not be forgotten to pay attention to it in proper time.)

In the second place or the tens we have then only a 4, instead of a 5 ; we are therefore again under the necessity of borrowing from the next higher figure, though this be an 0, subtracting then 7, from 14, the remainder 7, is written in the proper place. In the place of the hundreds we have then a 9, by the effect of the foregoing borrowing, from which the 2 subtracted gives the remainder 7, and the borrowing is now made from the place of the thousands. By the preceding borrowing, the 8, in the order of the thousands has now become 7, and is again insufficient to admit of a 9 being subtracted from it ; the borrowing of an unit of the higher order gives here 17, from which 9 being taken, gives 8 as remainder. The 9 in the next higher order has now, by the lending, become an 8, in order to subtract the 9 below from it, a unit of the next higher order is again borrowed, making it 18, subtracting 9 from it, gives 9, as the remainder to be written. The unit in the next higher order having been borrowed, the 0 remaining, is made into a 10, by borrowing an unit from the next higher order, from which 4 being subtracted, leaves 6. The next higher number being a 9 by the supposed borrowing from the higher order, and the same being the case for the next following 0, these two subtractions are made exactly like that in the hundreds; until ultimately the last left hand figure being a 6, higher than the number 3 of the subtrahend under it, the subtraction is possible; which being done, the number 243698778 presents the full remainder required by the subtraction, or is the difference between the two given numbers.

The proof of the correctness of this operation will again be found by the addition of the subtrahend and the remainder, which by carryings corresponding to the preceding borrowing, will again give in the ultimate sum the first or upper number, as seen by the example. Proper attention to the example here explained will teach how to act in every case that may occur in subtraction, and it will be proper for the scholar to be exercised upon a sufficient number of examples, that he may acquire facility in this operation.

§ 28. There are two other ways to perform this operation to obtain the same result; but the above explained course of reasoning is the one most closely connected with the nature of the question, and the implied requisites of the operation; it is therefore proper to keep the scholar to this consideration. When once he has gone through the whole course of arithmetic, he will easily see the two other methods, which if taught at this stage of the study would confuse his ideas, and are therefore intentionally omitted here.

Examples.—Give the value of

1st. 6,045 — 5,909 =
2d. 82,795 — 69,899 =

3d. 9,000,090 — 8,998,979 =
4th. 10,072 — 10,069 =
5th. 11,399 — 11,289 =
6th. 12,000,988 — 11,998,986 =
7th. 15,000,092 — 14,987,698 =
8th. 989,689 — 979,968 =
9th. 852,301 — 847,967 =
10th. 1,009,052 — 1,008,987 =
11th. 3,090,965 — 3,008,764 =
12th. 5,908,672 — 4,090,067 =
13th. 1,064,512 — 943,639 =

§ 29. MULTIPLICATION, as has been stated, is the addition of a given number repeated as many times as another number contains units, or indicates; thus every number is in itself the product of that number into the unit. It is indifferent which of the two numbers be considered as acting the one or the other part in the operation; therefore they are both equally called *Factors*; the result of the operation is called the *Product*.

It is necessary, in order to perform this operation with ease, in more complicated calculations, to commit to memory the product of the nine numbers expressed by our numerical symbols. It is needless for *written operations* to go any farther, because the higher multiplications overreach, in writing, our system of numeration, and thence do actually not come into use.

We have already seen that our system of numeration is a successive addition of the unit below 9; which being the last symbol of quantity, the next quantity is expressed by a change of place. If now we treat every one of the nine symbols in the same way, by the successive addition of itself; we obtain, successively, the product of each of these symbols in a similar manner, thus forming what is commonly called the multiplication table. Writing therefore the regular series of numbers as far as 9, in a horizontal line, add each of them to itself, writing the result under it; then to this sum add again the number at the head of the column, and so in succession, until the whole 9 symbols are exhausted, we shall have the following system of results:

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

Considering the preceding table, we find that the first column to the left, which again contains the series of natural numbers of our system of symbols, by the successive addition of the unit, keeps an account of all the other successive additions; or that it indicates how many times this addition has been repeated, and that the result of any number of such additions, of any one of the successive numbers, is always found in the meeting of the horizontal and vertical lines of the two numbers taken as factors; thus, for instance, under 7, and where the horizontal line marked 6, in the first column, meets it, we find 42, that is, the addition of 7 six times repeated gives the result 42. In like manner under 6, opposite to the 7 in the first column, will again be found 42. So 6 times 7, and 7 times 6, (such is the usual expression,) are equivalent; as has been stated above; and such is the case with any other number.

The regular progression of the different results is easily observable, and some attention to it will assist in fixing them in the memory; it is best not to load the beginner with a longer table, for which he has no use, until he may, in practical application, wish to calculate from memory, without writing, when the circumstance of its possessing interest and usefulness will make that task easy, which at this stage of instruction is a dry and useless labour.

§ 30. We must now suppose: that the scholar has acquired some facility in the use and application of the results of the preceding table; and shall proceed to show the details of multiplication by examples.

Be it given to multiply 358279 by 6; or to execute what is expressed by the sign of multiplication, thus: 6 × 358279.

Write the smaller factor, in this case the 6, under the other, so that the units stand under each other; then execute the multiplication of each of the numbers of the larger factor successively, and write the result under the horizontal line drawn below the factors, so that the right hand figure of the product shall always stand under the number multiplied, thus:

$$\begin{array}{rrrrrrr}
 & 3 & 5 & 8 & 2 & 7 & 9 \\
 & & & & & & 6 \\
\hline
 & & & & & 5 & 4 \\
 & & & & 4 & 2 & \\
 & & & 1 & 2 & & \\
 & & 4 & 8 & & & \\
 & 3 & 0 & & & & \\
1 & 8 & & & & & \\
\hline
2 & 1 & 4 & 9 & 6 & 7 & 4
\end{array}$$

then adding up all these products, the sum resulting will be the general product of the whole multiplication.

The inspection of this detailed execution of the preceding example, shows that we may again apply, in this case, the mode of abridgment that has been pointed out in addition, by carrying over to the next tenths without writing out the full results every time. We would, therefore, in the preceding example say, (analogous to what has been done in addition,) 6 times 9 is 54; write 4, and keep (or carry) 5; then keeping this 5 in mind, we would next say, 6 times 7, is 42, and 5, is 47; writing again the 7, and keeping the 4 to be added to the next product; then 6 times 2, is 12, and 4, is 16; when writing the 6, and keeping 1, and proceeding thus to the end of the number, we obtain at once the same numbers that appear above, in the final result. This mode of proceeding is therefore the usual mode of operating, with each of the numbers of the factor that is chosen, for the purpose of taking the multiples of the other by it; for which, as said before, it will be best to choose the smaller one, because it gives the shorter example in writing.

§ 31. When both factors are compound numbers, it is evident that the multiplication of each of the numbers of the one, cannot be made at once with all the numbers of the other; therefore we must proceed with each number of the one factor, exactly as shown above with the single number; and in order to give to each individual result its proper place, we must begin to write the first number of each product on the right hand side, exactly under the number of the multiplier of which it is the product; as its proper unit. The sum of all these partial products is then made, by the addition of all the numbers in the regular order in which they stand under each other, as this has been done in the preceding example, with the partial products of the simple number.

This will be shown in the following example, in which it is required to perform the multiplication 174392 × 6435; writing the factors properly under each other, so that the units stand under each other, and the other numbers follow in their regular order, the successive results in their proper places, will be as follows:

$$\begin{array}{r}
174392 \\
6435 \\
\hline
871960 \\
523176 \\
697568 \\
1046352 \\
\hline
1122212520
\end{array}$$

In this manner, every example, whatever quantity of figures it may be composed of, will stand

If any of the figures in the number to be multiplied, which is called the multiplicand, should be an 0, its product into any number whatsoever, is = 0; because 0 times any number whatever, always indicates that the number is not there; the place will, therefore, receive only that number which may be carried over from the multiplication of the preceding number, and if none be carried, only a 0.

If an 0, occur among the numbers by which the multiplication is to be performed, or the multiplier, the whole row of figures to be multiplied by it producing a result = 0, the place where the first number would stand will only be marked by an 0, and the multiplication by the next following number is begun in the same row, immediately after, thus placing each result in its proper place.

The following example will explain both the above cases, where the effect of the two 0's, in the multiplier is shown by the removal towards the left of the two latter rows of figures.

```
     3603904
       50203
    --------
    10811712
   72078080
180195200
------------
180926792512
```

§ 32. It will be proper to exercise the scholar in a variety of examples, until he has become accustomed to the operation, and is able to make any multiplication without error: the younger the scholar may be, the easier the examples must be in the beginning, and must gradually increase in difficulty, by the combination of different cases, in larger numbers. Still, in this it is to be observed: that when the beginner has performed examples gradually with the whole series of the nine simple numbers, it will be proper to show him only, what is the effect of a compound multiplier, as a repetition of the similar operation of one number only, and the addition of the different partial products into one whole; and not to follow servilely the augmentation by one number; (or place of figures,) that he may not, as often happens, consider that he has every time a new difficulty to overcome; but must himself come to the observation, that multiplication by a number of places of figures is a mere repetition of the operation he knows, requiring nothing but a little more attention, and more accuracy in the placing of the figures.

Examples.—Execute the following multiplications:

1st. 653 × 781 =
2d. 5032 × 683 =
3d. 749 × 903701 =
4th. 6094 × 75790838 =
5th. 89048 × 109373859 =
6th. 95638 × 9895076324 =
7th. 90184 × 1097685 =
8th. 340769 × 5946817 =
9th. 99987642 × 3246 =
10. 89736 × 65820934 =
11. 596 × 348962547 =
12. 978 × 804060789 =

§ 33. **DIVISION**, is an operation the opposite of *Multiplication*, as has already been stated; its problem is therefore: to find how many times a given number is contained in another given number, which is thus considered as a product of the first and the quantity sought.

The table of products, or multiplication table, given above, may therefore be here applied inversely; a ready and habitual knowledge of its results is therefore also constantly applied in this rule, by the comparison of its *results* with the quantities presenting themselves in an example.

While all the preceding operations have begun at the unit, this on the contrary must begin by the highest number, or order of symbols; for the greater number of times, which one quantity may be contained in another, is necessarily to be taken out, or considered, first, the inferior numbers will then follow in their regular order, and keeping account of the value of any remainder from the preceding operation in its proper rank, as in the following example, which we shall express in the manner that has been shown in section 20, in order to accustom the learner to retain the systematic language of the operation itself, which is always the preferable method, and maintains the necessary order in the calculation. With this view we shall draw a horizontal line under the dividend, under which we shall place the divisor, and the result, or quotient, will be written on the right hand side of the sign of equality which follows them, thus:

$$\frac{842316}{3} = \begin{array}{r} 280772 \\ 3 \\ \hline 842316 \end{array}$$

$$\begin{array}{l}
\;6 \\ \hline
\;24 \\
\;24 \\ \hline
\quad 23 \\
\quad 21 \\ \hline
\quad\; 21 \\
\quad\; 21 \\ \hline
\qquad 6 \\
\qquad 6 \\ \hline
\qquad 0
\end{array}$$

Here we say, 3 in 8, is contained twice, and having written the 2, as the first number, to the quotient, we must make the product of it by the divisor, write it under the corresponding number of the dividend, and subtract it from this; this product being 6, in this case, the subtraction of it from the first number of the dividend, the 8, leaves 2, as a remainder. Now, for the sake of easier distinction we place the next number by the right side of this remainder, which being 4, gives for the next number to be divided 24. Now 3, is in 24, contained 8 times; placing the 8 in the quotient, multiplying the 3 by it, the product of 3 times 8, placed under the 24, being also 24, leaves no remainder; placing the next number 2 down, we find, that 3 not being contained in it, we must indicate this by an 0, in the quotient, for the rank or order of the numeric system corresponding, which being done, the next number, 3, is taken down to the right side of the 2, which making 23, we say 3 in 23, will be contained 7 times; writing the 7 in the quotient, multiplying the 3 by it, and subtracting the product 21 from the 23, we obtain the remainder 2; taking down the 1 which gives 21, we say again, 3 in 21, is contained 7 times, and the product 3 times 7 being equal to 21, leaves no remainder; lastly, bringing down the 6, we find 3 in 6 twice, and writing the

2 in the quotient, and subtracting its product by 3, from the 6, leaves no remainder, and we obtain the exact quotient 280772.

Division being the opposite of multiplication, we have the means of proving this result, by the multiplication of the quotient by the divisor; the product of which must be equal to the dividend, as is evident from the definitions given of this operation.

Writing then the divisor under the quotient, and performing the multiplication, the product resulting will be equal to the dividend, if the whole operation has been rightly performed.*

§ 34. If the divisor is not contained an exact whole number of times in the dividend there will remain at the end of the division, a number smaller than this divisor, which is called the *Remainder*. In order to indicate fully the actual result of the division, this number is yet to be placed at the end of the quotient, with the divisor written under it, and a horizontal line between them, to indicate that this division should yet be made.

Such numbers as indicate a division which cannot be executed, are called *Proper fractions*, while every division, indicated as above, of a number larger than the divisor, is, in comparison with these, called an *Improper fraction*: and, when considered in this point of view, the number corresponding to the dividend, is called the *Numerator*, and the number corresponding to the divisor is called the *Denominator*; while the quotient, whatever it may be, will always represent the *Value* of the fraction.

---

* The divison can also be proved in other ways, as: the well known proof by 9, which is grounded upon the principles of the decimal system; the addition of the remainder and the different products of the individual quotients into the divisor, each in its proper rank, which are already written in the operation, and will of course produce again the dividend; and others. But it does not belong to this stage of instruction to deviate from the simple principles, to show *artifices* of calculation which either occur afterwards of themselves to an attentive calculator, or may be taught after a systematic course has been once gone through. By such lateral deviations from the straight simple course of the science, the ideas of the scholar are diverted from its simple system to mere accessories, to which he is apt to give an undue weight in his studies, and thereby lose the system of the science.

This general idea of fractions, the origin of which it is proper to show here, will hereafter be the fundamental idea from which the calculation of this kind of quantities is to be deduced.

The following is an example that will show such a division, and the mode of operating in the case.

Being given to divide

```
     7835921
     ───────  = 979490 1/8
        8              8
  ──────────    ──────────
     72          7835921
    ────
      63
      56
     ────
       75
       72
      ────
        39
        32
       ────
         72
         72
        ────
         01
```

In this example: we see that the first number of the highest order being smaller than the divisor, we must take it jointly with the next following number of the lower grade, and say: 8, in 78, is contained 9 times; and the 9 is written as the first number in the quotient; then making the product 8 × 9 = 72, it is written under the 78, from which it is subtracted, and leaves 6, which being written below the line, and the number next following in rank; the 3, written down to it, gives 63 for the next number, to be divided by 8, which being contained 7 times in it, 7 being written in the quotient, the product 7 × 8 = 56 is written under the 63, the subtraction performed, and the 5 or next following number placed down to the 7, that remains from the subtraction; the operation is thus continued, exactly as in the former example, until when the last number, 1, is set down at the side of the 0, we find that 8 is no longer contained in

1, and therefore write an 0 in the quotient; having no more numbers in the dividend, we find that 1 ought yet to be divided by 8, which we write in the quotient, as stated above, $\frac{1}{8}$ like an unexecuted division, or *a Proper fraction*.

When we make the proof of this example, as has been done in the preceding one, we consider the 1 as a remainder, and in the multiplication of the quotient by the divisor, add it to the product; so that we would here say 8 times 0, is 0, and the remainder 1 added, gives 1 for the first number of the product, exactly as in the dividend, and then continue the multiplication through the whole quotient obtained, as in the former example.

§ 35. When the divisor is a number composed of more than one figure, the principles of the operation remain the same; but it becomes necessary to pay attention to the effect of the multiplication of the quotient into the whole number of the divisor; which may render it necessary to take this quotient smaller than might appear from a mere comparison of the first numbers of the divisor and the dividend; all the rest of the operation is only an extension of the operations explained in the preceding examples; which have been described in detail, with the express view of giving a full explanation of the first elementary principles. Reasoning with the same details upon the following example, the operation of a division, with a divisor composed of more than one figure, will also be clear.

The following division being given

$$\frac{64059213}{758} = 84510\frac{633}{758}$$

6064	758
3419	676080
3032	422550
3872	591570
3790	633
821	64059213
758	
633	

Here in considering only the first number of the divisor, and comparing it with the two first of the dividend, we would find 7 in 64 contained 9 times; but we must take into consideration the multiples of the numbers which follow the hundreds in the divisor. The 5 tens, or 50, multiplied by 9 would give 45 tens, or 450, and 7 × 9 = 63, taken from 64, would leave only 1, which, considered as hundreds, as must be done in this case, would not allow us to take the 450 from it. We find, therefore, that the quotient 9, is too large. Taking 8, we find that 7 × 8 = 56, taken from 64 leaves 8 as remainder; and if we consider now the 58, as multiplied by 8, we find that the 4, which comes here again as hundreds to be subtracted from the 8 hundreds can be taken away with a considerable remainder. Writing then 8, as the first number in the quotient, we make the product 8 × 758, and place it under the respective numbers of the dividend, so that the product of the first number of the divisor, that is to say, 7 × 8 or 56, may stand under 64, and the other numbers follow in their regular order; we now make the subtraction, in the same manner as has been often before shown, which leaves 341 as remainder; as this is less than the divisor it also proves that no greater number could have been taken for the quotient; to this we join, as in the preceding examples, the number of the dividend next after those used in the last subtraction, which is here the 9 and present thus the total number 3419. We now proceed as before to compare the products of 7 with the 34, as the number presenting itself here for division, in the same rank as the 7 of the divisor; this shows 4 as the nearest factor producing with 7 a multiple, 28, inferior to 34, and leaving 6 as remainder, while 4 × 58 giving only a 2 to carry to the place of the hundreds, leaves sufficient room for the subtraction of the whole product; we thus obtain the remainder 387, that is again smaller than the divisor; and placing after it, the next following number of the dividend, the 2, we say first 7 into 38 is contained 5 times, and the product, 5 × 7 = 35, taken from 38 leaving 3, the product 58 × 5, giving only 2 to carry to the place of the hundreds, will leave a sufficient quantity for the subtraction; the 5 being placed in the quotient, and the substraction of its product performed, we have the remainder 82; then placing the 1 down after it, the resulting 821 contains the divisor, evidently, only once. Placing 1 in the quotient, the subtraction of 758 from 821 leaves 63; when the last figure, or 3, is written after this, the number 633, that results, being less than 758, the latter will not be contained in it; this gives an 0 in the quotient, for the last whole number; and the unexecutable division $\frac{633}{758}$ as a fraction or remainder; as in the second or foregoing example.

The proof of this example is again made in the same manner as in the last; multiplying 84510 × 758 and adding 633 to it, the dividend will again be obtained, as seen in the example.

The remark which has been already made, upon the propriety of practising any of the elementary operations until a competent dexterity is acquired, of course, also applies here.

The detailed manner shown here, is what is usually called *long division*; and even experienced calculators may often find it proper to apply it, when the number of places of figures in the divisor is great.

§ 36. For common calculation it is often desired to spare writing out the numbers for the subtraction, and writing only the remainders. This is carried on as in the following example.

Given $$\frac{9460753}{879} = 10763\frac{76}{879}$$

	879
6707	96867
5545	75341
2713	86104
76	76
	9460753

Here the divisor is contained once in the three first numbers of the dividend; the 1 being placed in the quotient, the subtraction is immediately made from them, and only the remainder placed below; which being 67, and the next number of the dividend, the 0, being put down to it, the divisor, 879, being larger than 670, the next number in the quotient becomes a 0. After writing it, the next number, the 7, is taken down from the dividend, and in the resulting 6707 the divisor is contained 7 times. Now the divisor is multiplied by 7, and the subtraction of the product made in the memory immediately, and again only the remainder written down; thus: say 7 times 9 is 63, subtracted from 67, which the number above must be supposed to represent, in order to allow the subtraction of the product of the unit or first number, leaves 4, which is written down as a remainder under the 7; the 6, which the number in the next higher rank has been supposed, is kept in memory, and added to the next higher order of numbers, with which it is then again subtracted; therefore, continuing the multiplication, we say: 7 times 7 is 49, the 6 kept being added makes 55, that subtracted from 60, which we suppose to be the number above, having the 0 in the first place to the right, the remainder, 5, is written under the 0, and 6 is kept to add to the next following product; for this we now say 7 times 8 is 56, and 6 carried is 62, taken from 67 leaves 5. Bringing now the 5 from the dividend down to the remainder 554, we have for our next dividend 5545, for which we say: 8 in 55 is contained 6 times; and as 6 × 8 = 48, leaves 7 in the place of the hundreds, for the carrying from the products of the lower numbers following it, the remainder is evidently large enough to allow the subtraction of the whole product; so we say again, 6 × 9 = 54, from 55, leaves 1; write it, and carry 5; then 6 × 7 = 42, and 5, is 47, from 54, leaves 7; write 7 down, and carry 5; lastly, 6 × 8 = 48, and 5 is 53, from 55, leaves 2; the remainder, presents therefore, 271; to which the 3, as next lower number in the dividend, being written, we find 7 in 27 is contained 3 times, or 3 × 7 = 21, leaves 6; a sufficient remainder in the hundreds, for the carrying of the product 3 × 79; so we say again, 3 × 9 = 27, from 33, leaves 6, and 3 to carry; then 3 × 7 = 21, and 3 added gives 24, from 31, leaves 7, and 3 to carry; then 3 × 8 = 24, and 3 is 27, which subtracts without a remainder, from the 27 above; and leaves the remainder 76, to which we have no other number to set down from the divisor, it therefore gives the numerator of the proper fraction remaining, thus $\frac{16}{573}$, is a division that cannot be executed with our present means.*

In the manner the reasoning has been carried, in this example, every other more complicated case is to be executed; it is therefore expected that it will suffice to introduce the learner into the practice of this method.

Examples.—Execute the following divisions:

1st. $\frac{34198}{72} =$

2d. $\frac{64059}{38} =$

3d. $\frac{980409}{84} =$

4th. $\frac{7260991}{96} =$

5th. $\frac{197403019}{107} =$

6th. $\frac{30409675}{3692} =$

7th. $\frac{85430072}{954} =$

8th. $\frac{186153042}{3964} =$

9th. $\frac{9860725814}{999} =$

10th. $\frac{3846721957}{4099} =$

11th. $\frac{1402376205}{90918} =$

12th. $\frac{904763825}{994703} =$

CHAPTER IV.

Of Vulgar Fractions.

§ 37. We have seen already, in section 21, and at the end of Division, that fractions are unexecuted divisions;

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* The proper fractions are still purposely here represented as unexecutable divisions, because the preceding operations in whole numbers, do not furnish any means for such a division. We shall

we have also seen, that in consequence of this, they consist of two parts, corresponding to the two parts or numbers engaged in a division; their form, or the manner of writing them, we have seen to arise naturally from the division, when a number remained ultimately in the dividend, which was smaller than the divisor, or the number by which it should be divided; we have there already observed, that this constituted a *Proper fraction*, while every division whatever, expressed in the same form, was an *Improper fraction*, as it would naturally be called, from its still containing the divisor a whole number of times.

The number above the horizontal line, (as seen in section 34,) which corresponds to the dividend, is called the *Numerator* of the fraction; and the number below this line, corresponding to the divisor, is called the *Denominator* of the fraction; thus considering the first as indicating the number of parts taken, and the second as indicating the value of the parts, or giving the name to the parts. By this means any fraction may evidently be represented as, (or rather these considerations show it to be actually) the product of a whole number, into unity divided by another number; this latter consideration characterizes them as a particular kind of quantities, in the same manner as the different places of figures characterize units, tens, hundreds, and so on: we thus evidently have (expressing the above reasoning according to the forms and signs adopted) for an example,

$$\tfrac{7}{18} = 7 \times \tfrac{1}{18}$$

where 7 is the numerator, counting the parts, and 18 the denominator, showing these parts to be eighteenths of the unit. And the value of these parts may evidently be as much varied as the numbers themselves; therefore they have not, like the numerical system, one necessary and uniform law of connexion.

§ 38. From these considerations of the principles and nature of fractions, the following three fundamental propositions for the arithmetic of fractions, naturally follow:

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afterwards show, how these values may be expressed, either exactly or approximately, by a continued division, and an extension of the decimal system, below the unit; that is to say, by decimal fractions.

Proposition i. *As many times as the Numerator of a fraction is made larger or smaller, the Denominator remaining unchanged, so many times the Value of the fraction is made larger or smaller.*

For: by multiplying the *numerator* by any number, there are *as many times more* parts taken as this number indicates; and in dividing it by any number, there are *as many times less* parts taken, as the number indicates; in the first case, therefore, the *value* of the fraction becomes *as many times larger*, and in the second, *as many times smaller*, as the number used in the multiplication, or division, indicates.

Example. $\frac{13 \times 7}{18} = 13 \times \frac{7}{18}$ according to the same reasoning as in the preceding section.

And $\frac{7 : 9}{18} = \frac{7}{18} : 9$, according to the same.

Or $\frac{9 \times 3}{16} = 9 \times \frac{3}{16}$ and $\frac{3 : 9}{16} = \frac{3}{16} : 9$

and so in every case.

Proposition ii. *As many times as the Denominator of a fraction is made larger or smaller, the Numerator remaining unchanged, so many times the value of the fraction is made smaller or larger.*

For: the *denominator* being the number by which the unit is divided, as many times as this number is multiplied, so many times the unit is divided into more parts; and therefore, the parts becoming as many times smaller, an equal number of them represents a value as many times smaller; that is to say, the value of the fraction is as many times smaller; and inversely, when the denominator is divided by a number, the unit is divided by a number as many times smaller as this divisor indicates; therefore, the parts become as many times larger; and the value of the fraction becomes as many times larger; all under the supposition: that an equal number of these parts be taken before and after the operation.

Example. $\dfrac{7}{18 \times 13}$ is 13 times smaller than $\dfrac{7}{18}$ because the 7 is divided by a number 13 times larger than 18; or we have, $7 \times \dfrac{1}{18 \times 13}$, 13 times smaller than $7 \times \dfrac{1}{18}$;

and $\dfrac{7}{18:9} = \dfrac{7}{2}$ is 9 times larger than $\dfrac{7}{18}$; because the 7 is divided into parts 9 times larger; or, we have, $7 \times \dfrac{1}{18}$, 9 times smaller than $7 \times \dfrac{1}{2}$

PROPOSITION III. *When the Numerator and Denominator of a fraction are both multiplied or divided by the same number, the Value of the fraction remains unchanged.*

This is an evident consequence of the combination of the two preceding propositions; they show the effect of the multiplication and division upon the numerator and the denominator, to be exactly opposite, and therefore, when performed with the same number, they exactly compensate each other; that is to say: *as many times* as the *Value* of the fraction becomes *larger or smaller*, by the *multiplication or division* of the *Numerator* of the fraction, so many times it becomes again *smaller or larger*, by the *multiplication or division* of the *Denominator*.

Example. $\dfrac{7 \times 9}{18 \times 9} = \dfrac{7}{18} = \dfrac{7:9}{18:9}$

where the mutual destruction of the effect, of the two operations, is self-evident.

The two first propositions solve directly all multiplication or division of fractions by whole numbers, in a double manner; for we have, evidently, every time, the choice between two operations, each of which may, according to the case, present a preference in the individual application.

The third proposition will evidently furnish us the means to reduce fractions from one denominator to certain other ones, in order to obtain the fractional parts expressed so as to be adapted to certain purposes in the operations of arithmetic, without changing their value.

§ 39. The investigations of section 37, have shown fractions to be equivalent to the product of a whole number into certain quantities expressed in parts of the unit; when thus representing quantities of different values or kinds, they have different denominators; their numerators therefore cannot be taken into one sum, or difference, without previous appropriate changes. By the third of the foregoing propositions, we have obtained means to make such changes, without altering the value of the fractions. The aim of such a change, must evidently be to obtain the same denomination for both, or all the fractions, whose sum or difference is desired.

We have seen in multiplication; that it is indifferent which of the two factors is multiplier or multiplicand, this shows that equal denominators may be obtained for two fractions, by multiplying the denominators together; if therefore, the numerators of the two fractions are also multiplied, each alternately by the denominator of the other, the value of the fraction will remain unchanged; according to the third proposition above; and if more fractions are concerned, considering the first result as one, and operating upon it in conjunction with another, exactly in the same way as before, and so on to the end, a result is evidently obtained, that applies to any number of fractions. This furnishes us with the following general rule.

To reduce fractions to a common denominator; multiply the numerator and denominator of each fraction by all the denominators except its own; then all the fractions will have the same denominator, and the numerators will be such that the value of the fractions will not be changed.

Example. $\frac{7}{15}$ and $\frac{3}{14}$ reduced to the same denominator

will give, $\frac{7\times14}{14\times15}$ and $\frac{3\times15}{14\times15}$; or $\frac{98}{210}$, and $\frac{45}{210}$;

where evidently the *Denominators*, that is the parts of the unit implied in the fractions, are the same.

Being given to reduce to the same denominator;

$$\frac{1}{2};\quad \frac{2}{3};\quad \frac{3}{5};\quad \frac{7}{8}$$

we evidently obtain, step by step, the following results: from the two first,

$$\frac{3}{2\times 3}\ \text{and}\ \frac{2\times 2}{3\times 2};\qquad \text{or}\ \frac{3}{6}\ \text{and}\ \frac{4}{6};$$

from these and the third:

$$\frac{3\times 5}{6\times 5};\quad \frac{4\times 5}{6\times 5};\quad \frac{3\times 6}{6\times 5};\quad \text{or}\ \frac{15}{30};\quad \frac{20}{30};\quad \frac{18}{30};$$

from these and the last,

$$\frac{15\times 8}{30\times 8};\quad \frac{20\times 8}{30\times 8};\quad \frac{18\times 8}{30\times 8};\quad \frac{30\times 7}{30\times 8};$$

$$\text{or}\ \frac{120}{240};\quad \frac{160}{240};\quad \frac{144}{240};\quad \frac{210}{240};$$

Here quantities of the same kind, are evidently obtained; say equal parts of the unit, only in different quantities; such as correspond to the *new numerator*, and produce no change in the *Value* of the fraction, according to the principles stated; for, according to what has been seen above, these fractions might be thus written:

$$120\times\frac{1}{240};\quad 160\times\frac{1}{240};\quad 144\times\frac{1}{240};\quad 210\times\frac{1}{240};$$

Examples.—Reduce the following fractions to a common denominator.

1. $\frac{1}{6};\ \frac{9}{11};\ \frac{5}{7};\ \frac{3}{8};$

2. $\frac{6}{9};\ \frac{2}{8};\ \frac{3}{5};$

7. $\frac{5}{8};\ \frac{3}{10};\ \frac{4}{9};\ \frac{7}{12};\ \frac{9}{11},$

8. $\frac{1}{4};\ \frac{3}{9};\ \frac{1}{3};\ \frac{1}{5};\ \frac{7}{9};$

3. $\frac{2}{5}$; $\frac{3}{8}$; $\frac{7}{9}$;

4. $\frac{4}{7}$; $\frac{5}{9}$: $\frac{6}{11}$; $\frac{3}{8}$;

5. $\frac{9}{10}$; $\frac{2}{7}$; $\frac{5}{9}$; $\frac{4}{5}$;

6. $\frac{3}{7}$; $\frac{2}{9}$; $\frac{7}{15}$; $\frac{1}{6}$;

9. $\frac{3}{11}$; $\frac{5}{9}$; $\frac{7}{8}$; $\frac{9}{10}$; $\frac{19}{24}$;

10. $\frac{5}{6}$; $\frac{7}{9}$; $\frac{3}{4}$; $\frac{2}{5}$; $\frac{5}{9}$;

11. $\frac{1}{7}$; $\frac{3}{11}$; $\frac{5}{6}$; $\frac{9}{14}$; $\frac{6}{7}$; $\frac{13}{14}$;

12. $\frac{4}{5}$; $\frac{6}{7}$; $\frac{7}{8}$; $\frac{8}{9}$; $\frac{9}{10}$; $\frac{10}{11}$;

§ 40. It is evident from the above, that fractions cannot be reduced to any denominator indiscriminately, if as it is proposed in common arithmetic, the numerators or denominators shall not themselves contain fractions; that therefore the new denominator must be a multiple or a quotient, of the former denominator.

If it should become necessary to take whole numbers under the same consideration, it will easily be judged, from what has been said, that they must be considered as having the denominator, 1, and such indeed they are, for the unit is their measure, as to value; like any other denominator in a fraction,

Example. $34 = \frac{34}{1} = 34 \times \frac{1}{1}$;

For: every whole number whatsoever, must be considered as multiplied by 1, really to be a quantity; if it was multiplied by 0, it would be said not to be at all, as 0, denotes the absence of all quantity; and if multiplied by any other number, the product would be another number, corresponding to this multiplier.

§ 41. The continued multiplication of all the denominators evidently leads into large numbers, both for the numerators and the denominators, which it is desirable to avoid wherever possible; this will be the case when some of the denominators are products of the same number with different numbers, or have what is called, *Common factors*; these are therefore not necessary to be repeated in the continued product of the denominators, which furnishes the new denominator, as the above example already shows, where 2 and 8, are products of 2, the first by 1, the second by 4.

The following problem and its solution, which will best be explained immediately by an example, will lead to this result.

PROBLEM. *To find the smallest number which will be divisible by several other given numbers.*

Solution. Write the numbers after each other,

$$\begin{array}{lccccccc|c} \text{as :} & 3\ ; & 4\ ; & 9\ ; & 10\ ; & 21\ ; & 35\ ; & 12 & \\ & 1\ , & 4\ , & 3\ , & 10\ , & 7\ , & 35\ , & 4 & 3 \\ & 1\ , & 1\ , & 3\ , & 10\ , & 7\ , & 35\ , & 1 & 4 \\ & 1\ , & 1\ , & 3\ , & 2\ , & 7\ , & 7\ , & 1 & 5 \\ & 1\ , & 1\ , & 3\ , & 2\ , & 1\ , & 1\ , & 1 & 7 \end{array}$$

take any one number, which will divide several of these numbers without remainder, and divide these numbers by it, write the divisor, here 3, on the other side of a vertical line; the quotient obtained under each of the numbers; write also all the other numbers, that are not divisible, down in the line, (as shown here in the second line of figures,) with the quotients; with this new series of numbers proceed as before; here we find the common divisor 4, and the third line of numbers is obtained; this line is reduced by the divisor 5, and the fourth row of figures is obtained; thus the operation is continued in the same way, until no common divisor is found; as in the fifth line of the example. The continued product of these quotients and the remaining number, which is here

$$3 \times 2 \times 7 \times 5 \times 4 \times 3 = 2520$$

will be the smallest number divisible without remainder by all the given numbers. The units of course disappear in the multiplication, as they do not augment the product; they indicate the number of reductions obtained by the operation, without which the continued product would have been $= 9525600$; and these two numbers are both equally divisible by the numbers first given. The proof of this operation lies in this: that those factors that have disappeared, being only such as were repeated in the given numbers, by these being different multiples of them, the division of the number obtained by the numbers first given, will always give a whole number ; thus are obtained in the example the following numbers :

$$\frac{2520}{3} = 840; \quad \frac{2520}{4} = 630; \quad \frac{2520}{9} = 280; \quad \frac{2520}{10} = 252;$$

$$\frac{2520}{21} = 120; \quad \frac{2520}{35} = 72; \quad \frac{2520}{12} = 210;$$

which all present quotients without any remainder.

§ 42. If therefore fractions, having the above numbers for their denominators, were to be brought under the same denominator, the quotients arising from the division of the new general denominator by all the first denominators successively, will give for each of the fractions the number by which it is to be multiplied in numerator and denominator, to reduce it to that common denominator ; thereby to furnish the means to obtain a new series of fractions, equivalent to the original ones, and all having the same denominator. The following fractions would therefore be changed, as presented by the following operation :

$$\frac{1}{3}; \quad \frac{3}{4}; \quad \frac{2}{9}; \quad \frac{7}{10}; \quad \frac{4}{21}; \quad \frac{8}{35}; \quad \frac{5}{12};$$

$$\frac{1 \times 840}{2520}; \quad \frac{3 \times 630}{2520}; \quad \frac{2 \times 280}{2520}; \quad \frac{7 \times 252}{2520}; \quad \frac{4 \times 120}{2520};$$

$$\frac{8 \times 72}{2520}; \quad \frac{5 \times 210}{2520};$$ that is

$$\frac{840}{2520}; \quad \frac{1890}{2520}; \quad \frac{560}{2520}; \quad \frac{1764}{2520}; \quad \frac{480}{2520}; \quad \frac{576}{2520}; \quad \frac{1050}{2520};$$

Thus the fractions are all brought to present equal parts, of a denomination inferior to the continued product of the original denominators, and capable of being added, or subtracted like whole numbers by their numerators only.

§ 43. **ADDITION OF FRACTIONS.** By the preceding sections the fractions have been brought to a shape which admits their being added and subtracted like whole numbers, as these operations have shown how fractions can be made to present the same parts, or to be quantities of the same kind, without changing their value; thus the rule to execute an addition of fractions, is now easily deduced, as follows:

Reduce the fractions to a common denominator, add the resulting new numerators, and give the sum the new denominator.

1st. *Example.* Find the single fraction corresponding to

$$\frac{1}{5}+\frac{7}{8}\text{; this gives: }\frac{8}{5\times 8}+\frac{7\times 5}{5\times 8}=\frac{8}{40}+\frac{35}{40}=\frac{43}{40}$$

$$=1+\frac{3}{40}$$; if the division is executed as it can be done.

2nd. *Example.*

$$\frac{2}{3}+\frac{7}{9}=\frac{2\times 9}{3\times 9}+\frac{3\times 7}{3\times 9}=\frac{18+21}{27}=\frac{39}{27}=1+\frac{12}{27}$$

Here the numerator and denominator of the fractional part both admit of division, by 3, and the sum becomes by it, $= 1 + \frac{4}{9}$.

This application of the third proposition of section 38, is to be made, whenever admissible, at the end of any operation upon fractions; because fractions are always to be presented in their lowest denomination.

3d. *Example.* Suppose the fractions given to be added, upon which the reduction to the same denominator has been performed in the preceding section. The following will be the results successively; being given

$$\frac{1}{3}+\frac{3}{4}+\frac{2}{9}+\frac{7}{10}+\frac{4}{21}+\frac{8}{35}+\frac{5}{12}$$

and making the sum of the new numerators obtained before, we have the following addition of whole numbers to make,

$$\begin{array}{r} 840 \\ 1890 \\ 560 \\ 1764 \\ 480 \\ 576 \\ 1050 \\ \hline 7160 \end{array}$$

giving the total fractional sum, $\frac{7160}{2520}$ which being an improper fraction, gives $2 + \frac{2120}{2520}$; the fractional part is reducible, as follows, by 40,

as $\frac{2120}{2520}\Big|\overset{40}{\frac{53}{63}}$ therefore, the ultimate sum is $= 2 + \frac{53}{63}$

Examples.—Execute

1st. $\frac{7}{8} + \frac{3}{7} + \frac{5}{9} + \frac{3}{14} + \frac{9}{24} + \frac{12}{15} =$

2nd. $\frac{9}{11} + \frac{2}{5} + \frac{7}{9} + \frac{8}{11} + \frac{3}{25} + \frac{9}{32} =$

3d. $\frac{1}{5} + \frac{7}{8} + \frac{2}{9} + \frac{17}{27} + \frac{19}{32} + \frac{15}{38} + \frac{16}{42} =$

4th. $\frac{9}{13} + \frac{15}{19} + \frac{13}{21} + \frac{14}{27} + \frac{8}{23} + \frac{10}{34} + \frac{5}{18} =$

5*th*. $\frac{7}{9}+\frac{6}{11}+\frac{3}{16}+\frac{5}{12}+\frac{9}{13}=$

6*th*. $\frac{1}{5}+\frac{2}{7}+\frac{3}{10}+\frac{4}{11}+\frac{5}{12}+\frac{6}{13}=$

7*th*. $\frac{9}{11}+\frac{12}{13}+\frac{13}{14}+\frac{5}{16}+\frac{8}{15}+\frac{7}{13}=$

8*th*. $\frac{2}{13}+\frac{5}{18}+\frac{6}{17}+\frac{3}{19}+\frac{4}{23}+\frac{5}{29}=$

9*th*. $\frac{7}{8}+\frac{9}{14}+\frac{3}{17}+\frac{6}{25}+\frac{21}{25}+\frac{1}{9}=$

10*th*. $\frac{2}{5}+\frac{3}{8}+\frac{2}{11}+\frac{7}{16}+\frac{8}{19}+\frac{2}{13}=$

11*th*. $\frac{5}{8}+\frac{3}{10}+\frac{4}{9}+\frac{7}{12}+\frac{9}{11}=$

12*th*. $\frac{1}{7}+\frac{3}{11}+\frac{5}{6}+\frac{9}{14}+\frac{6}{13}=$

§ 44. SUBTRACTION OF FRACTIONS. This differs from their addition only in the second part, as may easily be inferred from all the preceding reasoning; we obtain therefore the rule: *Reduce the fractions to a common denominator, subtract the new numerators from each other, and give to the remainder the new denominator.*

The proof of this rule is evident; by bringing the fractions to the same denominator, in the same manner as shown in the addition, the operation is reduced to the subtraction of the whole numbers, expressing the numerators, as is done in the addition.

Example. To subtract as indicated here:

$$\frac{7}{9}-\frac{2}{3}=\frac{3\times7}{3\times9}-\frac{2\times9}{3\times9}=\frac{21}{27}-\frac{18}{27}=\frac{3}{27}=\frac{1}{9}.$$

2nd. Example.

$$\frac{7}{8}-\frac{1}{5}=\frac{7\times5}{5\times8}-\frac{8}{5\times8}=\frac{35}{40}-\frac{8}{40}=\frac{27}{40};$$

Which process is evident by mere inspection, compared with the rules, and supported by all the preceding reasoning.

Examples.

1st. $\frac{7}{9}-\frac{13}{16}=$

2nd. $\frac{2}{5}-\frac{3}{8}=$

3d. $\frac{6}{7}-\frac{3}{14}=$

4th. $\frac{4}{5}-\frac{2}{11}=$

5th. $\frac{7}{8}-\frac{2}{3}=$

6th. $\frac{4}{7}-\frac{1}{3}=$

7th. $\frac{1}{2}-\frac{4}{9}=$

8th. $\frac{9}{10}-\frac{2}{5}=$

9th. $\frac{8}{9}-\frac{7}{16}=$

10th. $\frac{5}{6}-\frac{2}{5}=$

11th. $\frac{6}{11}-\frac{2}{19}=$

12th. $\frac{7}{27}-\frac{3}{37}=$

§ 45. The total value of a number of fractions, of which some are to be added, and others to be subtracted, may thus be taken in one, and under the smallest denominator; with this difference only: that a separate sum is to be made of all the new numerators to be added, and all those to be subtracted, and the sum of these latter to be subtracted from the former, for the new numerator.

The following example will show the process.

$$\frac{1}{3}+\frac{1}{6}-\frac{3}{5}+\frac{7}{15}+\frac{6}{11}-\frac{4}{27}+\frac{5}{18}-\frac{8}{25}$$

being given; find the smallest number divisible by all the denominators thus:

$$\begin{array}{cccccccc|c}
3, & 6, & 5, & 15, & 11, & 27, & 18, & 25 & \\
1, & 2, & 5, & 5, & 11, & 9, & 6, & 25 & 3 \\
1, & 2, & 1, & 1, & 11, & 9, & 6, & 5 & 5 \\
1, & 1, & 1, & 1, & 11, & 9, & 3, & 5 & 2 \\
1, & 1, & 1, & 1, & 11, & 3, & 1, & 5 & 3
\end{array}$$

the new denominator is,

$$= 11 \times 3 \times 5 \times 3 \times 2 \times 5 \times 3 = 14850$$

the successive multipliers of the fractions are,

$$\frac{14850}{3} = 4950\,; \quad \frac{14850}{6} = 2475\,; \quad \frac{14850}{5} = 2970\,;$$

$$\frac{14850}{15} = 990\,; \quad \frac{14850}{11} = 1350\,; \quad \frac{14850}{27} = 550\,;$$

$$\frac{14850}{18} = 825\,; \quad \frac{14850}{25} = 594\,;$$

Forming now the new numerators, by multiplying the old ones by their respective numbers, just found, and bringing those that are to be added into one column, and those that are to be subtracted into another column, then taking their difference for the resulting numerator, we obtain:

$$\begin{array}{rr}
+\,4950 & -\;\;8910 \\
2475 & 2200 \\
6930 & 4752 \\
8100 & \overline{} \\
4125 & 15862 \\
\hline
+\,26580 & \\
-\,15862 & \\
\hline
10718 &
\end{array}$$

the fraction resulting is therefore :

$$\frac{26580 - 15862}{14850} = \frac{10718}{14850}$$

This fraction may be still further reduced ; the mode of doing this at once to the greatest extent, by finding the greatest common divisor of the numerator and the denominator, will be shown hereafter ; in the example the division by 2 is evidently admissible, and we obtain by it,

$$\frac{10718}{14850}\,\overset{2}{\Big|}\,\frac{5359}{7425}$$

Though we had, in the above examples, taken the smallest number divisible by all the denominators, the ultimate fraction was still reducible ; this arises from the individual circumstance of the resulting numerator being such as to have a multiplier common with the denominator, in the same manner as the denominators, first given, had.

Examples

1. $\frac{3}{7} - \frac{2}{9} + \frac{7}{15} + \frac{1}{6} - \frac{3}{8} =$

2. $\frac{8}{25} - \frac{5}{8} + \frac{3}{10} - \frac{4}{9} + \frac{7}{12} - \frac{9}{11} + \frac{11}{12} =$

3. $\frac{1}{4}-\frac{3}{8}+\frac{1}{3}-\frac{1}{5}+\frac{7}{9}-\frac{3}{11}=$

4. $\frac{5}{9}+\frac{7}{8}-\frac{9}{10}+\frac{19}{24}-\frac{7}{16}+\frac{5}{6}+\frac{7}{9}=$

5. $\frac{3}{4}-\frac{2}{5}+\frac{5}{9}-\frac{1}{7}+\frac{3}{11}-\frac{5}{6}+\frac{9}{14}+\frac{5}{6}=$

6. $\frac{3}{7}-\frac{4}{5}-\frac{3}{11}+\frac{7}{12}+\frac{9}{14}-\frac{3}{8}+\frac{6}{35}+\frac{11}{12}-\frac{2}{15}$

7. $\frac{4}{7}-\frac{2}{9}+\frac{3}{14}-\frac{2}{15}+\frac{5}{8}+\frac{5}{12}-\frac{7}{16}+\frac{11}{18}-\frac{4}{5}$

8. $\frac{3}{8}+\frac{4}{5}-\frac{2}{9}-\frac{7}{15}+\frac{3}{14}-\frac{6}{21}+\frac{6}{25}-\frac{9}{26}+\frac{1}{3}=$

§ 46. MULTIPLICATION OF FRACTIONS

The close connection of the subject of fractions, considered as unexecuted divisions, with division; and consequently with its opposite, multiplication, renders the operations of multiplication and division of fractions more simple than their addition and subtraction.

For their multiplication the rule is simply,

Multiply the numerators into the numerators, and the denominators into the denominators; the resulting fraction will be the product of the fractions multiplied.

The proof of this rule lies in the first two elementary propositions upon fractions, stated in section 38; for; by multiplying a fraction by the numerator of another, this fraction has been made as many times larger as the numerator employed indicates; but as it was required to multiply it by a number, as many times smaller than this number, as the denominator of the fraction, whose numerator has been employed, indicates, the multiplication of the denominator by the denominator of that fraction makes the value of the resulting fraction just again as many times smaller; as is required.

Example. To multiply the fractions $\frac{7}{8}$ and $\frac{3}{10}$; into each other; or, to execute $\frac{7}{8} \times \frac{3}{10}$; the multiplication of $\frac{7}{8}$ by 3 gives $\frac{3 \times 7}{8} = \frac{21}{8}$; multiplying then the denominator of $\frac{21}{8}$ by 10, or making $\frac{21}{8 \times 10} = \frac{21}{80}$; the result is the value of the multiplication desired.

In like manner the following result is obtained:

$$\frac{3}{8} \times \frac{6}{11} = \frac{3 \times 6}{8 \times 11} = \frac{18}{88} = \frac{9}{44};$$

this last by reducing the fraction, by the division of the numerator and denominator by 2.

Examples.

1st. $\frac{3}{7} \times \frac{5}{8} =$ *5th.* $\frac{5}{8} \times \frac{3}{5} \times \frac{9}{10} =$

2nd. $\frac{4}{5} \times \frac{3}{11} =$ *6th.* $\frac{2}{5} \times \frac{3}{7} \times \frac{4}{9} =$

3d. $\frac{2}{3} \times \frac{6}{5} =$ *7th.* $\frac{3}{5} \times \frac{4}{7} \times \frac{5}{9} =$

4th. $\frac{6}{11} \times \frac{7}{9} =$ *8th.* $\frac{4}{11} \times \frac{6}{13} \times \frac{7}{19} =$

9th. $\frac{1}{5} \times \frac{3}{8} \times \frac{7}{9} =$ 10th. $\frac{8}{13} \times \frac{7}{15} \times \frac{8}{8} \times \frac{2}{9} =$

§ 47. DIVISION OF FRACTIONS. According to the principles and propositions presented in the beginning, division may evidently be performed by dividing the numerator of the dividend by the numerator of the divisor, and the denominator of the dividend by the denominator of the divisor. But as this operation would often give fractional results for the new numerator and denominator, it is not employed; and the principle: that division is the inverse of multiplication, is here made use of, in concordance with the two first propositions respecting fractions, shown in section 38, from which is deduced the following rule:

Multiply the numerator of the dividend by the denominator of the divisor, and the denominator of the dividend by the numerator of the divisor; the first gives the numerator, the second the denominator of the resulting fraction; or the quotient.

To prove this, we need only invert the reasoning used in Multiplication; by multiplying the denominator of the dividend by the numerator of the divisor, the fraction has been made as many times too small, as the denominator of the divisor indicates; and by the multiplication of the numerator of the dividend by the denominator of the divisor, the value of the fraction is again made as many times larger; so that the ultimate result presents the real value of the quotient.

Ex. To divide $\frac{4}{5}$ by $\frac{3}{8}$; or to execute $\frac{4}{5} : \frac{3}{8}$; we obtain, by the multiplication, of the denominator of the dividend by the numerator of the divisor $\frac{4}{5 \times 3}$ three times smaller than $\frac{4}{5}$, and eight times too small; multiplying therefore the numerator of this result by 8, the denominator of the divisor, we obtain,

$$\frac{4\times 8}{5\times 3}=\frac{32}{15}=2+\frac{2}{15}$$

this last again, like in a former case by dividing the numerator by the denominator and keeping only the remainder as a fraction, as the result presents again an improper fraction; that is to say, the fraction

$\frac{3}{8}$ is contained $\left(2+\frac{2}{15}\right)$ times in the fraction $\frac{4}{5}$.

It is evident, that the result of such a division may give a whole number, as well as the division of whole numbers may give; for a fraction can be contained a whole number of times in another fraction, as one whole number in another. As for example:

$$\frac{3}{4}:\frac{1}{8}=\frac{3\times 8}{4}=\frac{24}{4}=6$$

We may also proceed by the principle of the third proposition alone; namely, the reduction of a fraction without changing its value. For that we must write the intended division fully out in the form of a fraction, in numerator and in denominator; so the example above would stand,

$$\frac{4}{5}:\frac{3}{8}=\frac{\frac{4}{5}}{\frac{3}{8}}$$

When we multiply here, the numerator and denominator, by the denominators of the individual fractions, we shall compensate the divisors or denominators of the individual fractions, and have the numerators affected, alternately, the one by the multiplication of the denominator of the other thus:

$$\frac{\dfrac{8\times4}{5\times8}}{\dfrac{3\times5}{5\times8}}=\frac{8\times4}{3\times5}=\frac{32}{15}=2+\frac{2}{15}$$

For in bringing the fractions to the same denominator, these denominators of course compensate in numerator and denominator, as shown in the third Proposition.

Examples.

1st. $\frac{5}{8} \div \frac{3}{11} =$

2nd. $\frac{4}{5} \div \frac{8}{15} =$

3d. $\frac{7}{10} \div \frac{1}{8} =$

4th. $\frac{9}{11} \div \frac{5}{9} =$

5th. $\frac{6}{7} \div \frac{4}{9} =$

6th. $\frac{9}{16} \div \frac{4}{15} =$

7th. $\frac{8}{11} \div \frac{1}{9} =$

8th. $\frac{2}{9} \div \frac{2}{13} =$

9th. $\frac{9}{10} \div \frac{8}{13} =$

10th. $\frac{11}{12} \div \frac{3}{5} =$

§ 48. It is proper here to add some remarks upon the manner of proceeding in certain cases, to facilitate the calculation of fractions, as much useless and tedious calculation may be avoided by some attention to the relation of the numbers given, and the reductions which they may thereby present; this operation, by easing the calculation, will also make it less liable to mistakes.

First. If any given fraction is not reduced to its simplest expression, it is proper to reduce it previous to performing the operations required; as for example:

It being given to make $\frac{3}{12} + \frac{4}{20} + \frac{6}{18}$, the fractions are immediately to be reduced by the equal divisions of numerators and denominators, that are evidently possible, to the following:

$$\frac{1}{4} + \frac{1}{5} + \frac{1}{3}$$

which are then to be added according to the principles above given.

Second. In the multiplication or division of fractions it may occasionally occur, that such multiplications or divisions as would compensate each other in the ultimate result, may be avoided by some attention, and that such advantage may be taken of the relation of the numbers that may at once effect a reduction by a division of the one term, instead of a multiplication of the other; as for instance:

$\frac{3}{8} \times \frac{4}{9}$; the denominators of each being respectively multiples of the numerators of the other, the multiplication is useless, and the division may be made alternately, the 4 being contained twice in 8, and the 3 thrice in the 9; so that it can be written immediately,

$$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

or in division for example; given $\frac{3}{5} : \frac{6}{25}$ which inverted into a multiplication as, $\frac{3}{5} \times \frac{25}{6}$; evidently presents, for the same reason as before, $\frac{1}{2} \times 5 = \frac{5}{2} = 2 + \frac{1}{2}$

§ 49. **REDUCTION OF FRACTIONS.** This must be done according to the principles of the third proposition of section 38, and may be effected: either by successively dividing in numerator and denominator, by such numbers as are observed to divide them both without remainder; or at once, finding first the *greatest common measure* between them; as it is evident that the greatest divisor must produce at once the greatest possible reduction.

The former may often be easy for an experienced calculator; it is for instance easily seen that all even numbers are divisible by 2; all those ending by 0, or 5, by 5; the division by 3 is often easily discovered; &c.

So for example may be done with the following fraction, where the successive divisors are marked above the partition line between the successively reduced fractions.

$$\frac{1260}{2940}\overset{10}{\Big|}\frac{126}{294}\overset{2}{\Big|}\frac{63}{147}\overset{7}{\Big|}\frac{9}{21}\overset{3}{\Big|}\frac{3}{7}$$

or.

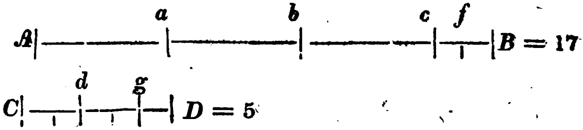
$$\frac{1296}{3888}\overset{3}{\Big|}\frac{432}{1296}\overset{2}{\Big|}\frac{216}{648}\overset{9}{\Big|}\frac{24}{72}\overset{4}{\Big|}\frac{6}{18}\overset{6}{\Big|}\frac{1}{3}$$

To make the reduction at once the greatest possible, requires at first to find the greatest number that will divide both numerator and denominator without remainder, which is what has just been called the greatest common measure, thence the solution of the following

PROBLEM. *To find the greatest common measure of two given numbers.*

In order to present the principles of this operation in the clearest light, it is best to represent the two quantities as two linear dimensions; this may evidently be done, as any quantity may be represented by a line containing as many units of measure, as the number itself contains units of abstract quantity.

Let therefore the two lines AB, and CD, represent the two numbers,



It is clear that there can be no greater number that will divide the two numbers, (or no greater line dividing the two lines,) without a remainder, than the smaller number (or line) itself; dividing therefore the greater number by the smaller, (or taking the smaller line from the greater, as often as possible,) as in the above $CD = Aa = ab = bc$, or three times, the cB remains, smaller than CD; which presents therefore the remainder not divisible by CD, and smaller than it.

But the number which can divide both AB and CD, without remainder, must also divide this remainder cB; it must at the same time also divide CD, as in that case it would divide its equal Aa, ab, and bc; therefore, between these two cB and CD; the same reasoning, used above, applies again; namely: that they can have no greater common measure than the smaller, or cB, itself; therefore, divide CD, by cB; let us suppose, that as in the figure it is contained twice in it, or $cB = Cd = dg$, and leaves again the remainder gD, smaller than cB; between this remainder and the former divisor, or the gD, and the cB, the same reasoning takes place as before, their greatest possible common measure could only be the gD, itself; dividing therefore the cB, by the gD, and supposing it is contained exactly twice in it, or $gD = cf = fB$, and that it leaves no remainder; then this gD, that is the last divisor, will be the greatest common measure possible, between the two numbers represented by AB, and CD; (it may be observed, that this operation is to be continued, as long as a remainder is obtained by these successive divisions.)

Because the gD, measures the cB, without remainder, and this cB measures the Cg, the gD measures also the CD; the Ac being a multiple of CD, is therefore also measured by gD, and the other part of AB, namely cB, being also measured by it, the whole AB is measured by gD, which measures also CD; therefore, it is their *common measure*, and as we have always proceeded by the *greatest number*, which possibly could divide the two numbers successively given, it is also the *greatest common measure*, as was required.

If no divisor is found, except the unit, the two numbers have no greater common measure; that is, they are *Prime* to each other.

To apply this to numbers, let the fraction given be the following;

$\frac{45}{153}$ dividing the denominator by the numerator, the last being always the smaller number in a proper fraction, (of which alone there can here be question, because an improper fraction must first be reduced, by dividing by its denominator) we make the result of

$\frac{153}{45} = 3 + \frac{18}{45}$ that is: we obtain the first quotient 3, and the remainder $\frac{18}{45}$; this fraction inverted for the similar division, and the division executed gives $\frac{45}{18} = 2 + \frac{9}{18}$; or the quotient 2, and the remainder $\frac{9}{18}$; which treated as above, gives $\frac{18}{9} = 2$, as last quotient without a remainder; and proves the last divisor 9, to be the *greatest common measure*; and in fact we find that 9 is in 45 five times; in 153, seventeen times; so this applied to the figure would say, that, gD, is equal to nine of the units used in the measurement of AB, and CD. From this we obtain for the operation of the greatest reduction of the fraction,

$\frac{45 : 9}{153 : 9} = \frac{5}{17}$; which can be no farther divided, or reduced.

In the usual mode of writing, this example would stand thus:

45)153(3
18)45(2
9)18(2

Examples.

1*st.*	$\frac{24598}{44226}$	2*nd.*	$\frac{75844}{150579}$	3*d.*	$\frac{32}{658}$
4*th.*	$\frac{42986}{72598}$	5*th.*	$\frac{61047}{77373}$	6*th.*	$\frac{46}{978}$
7*th.*	$\frac{2209}{15063}$	8*th.*	$\frac{465}{1302}$	9*th.*	$\frac{693}{1815}$

§ 50. If by the foregoing process, no number is found dividing any one of the remainders, successively resulting, without a remainder, except the unit, or, the numbers have no common measure; the fraction is not *exactly* reducible into smaller numbers.

It is however evident, by the foregoing process: that by the successive division, the remainders have become successively smaller; and we might say, in respect to a given case, always less important.

If the above operation had been interrupted at any one of the steps, it is clear, that the part neglected, would have been a *fraction of the last subdivision*, made by the division of the last quotient by the last remainder; therefore, so much the smaller, the farther this division has been carried, and thence the influence of this neglected part upon the *Value* of the fraction so much the less. Considering this fraction as only an unit, having the last quotient for a divisor, and the preceding quotient as the whole number of these quantities, to which this fraction belonged, we shall have these reduced to the improper fraction of that denomination, by multiplying this number with the denominator, and adding the numerator, that is the unit; then considering this result again as the denominator or divisor, multiplying the next preceding quotient with it, and adding the number obtained last as the numerator of the corresponding fraction, and thus continuing the same process to the beginning, through all the quotients obtained, adding always to the successive products the numerators last obtained, we shall ultimately have the numerator and the denominator of a fraction, approaching to the fraction which is intended to be approximated, as near as the divisions executed will admit; that is with the neglect of only that fraction of the last subdivision which has been neglected, as above stated; for if the division was carried fully to the end, we would again obtain the full value of the fraction, as shown above. And, as it appears by the order in which the division has been made, namely, the inversion of the numerator into a divisor, and the denominator into a dividend, the last number resulting from this continued multiplication will be the denominator of the approximate fraction, and that obtained immediately before will be the numerator. This operation may be expressed by the following rule:

Divide the denominator by the numerator, and the last divisor by the remainder; always marking the quotient, as far as the approximation is intended to be carried; (as in finding the greatest common measure;) *then from the place where the operation is interrupted, make the continued product of all the quotients, adding unity to the first product, and afterwards always the last previous result, until the first quotient is arrived at. The last number resulting will be the denominator of the approximate fraction, and the one obtained immediately before, the numerator.*

Example. Let the fraction $\frac{98215}{367459}$ be approximated: the successive division will give,

```
98215)367459(3
         72814)98215(1
                  25401)72814(2
                           22012)25401(1
                                    3389)22012(6
                                             1678)3389(2
```

The successive approximations will be:

1st approximation. = 3 : 1, or the fraction $\frac{1}{3}$

2d " successive quotients, 3, 1, 2
continued products, 11, 3, 2, 1, or $\frac{3}{11}$

3d " successive quotients, 3, 1, 2, 1
continued products, 15, 4, 3, 1, 1, or $\frac{4}{15}$

4th " 3, 1, 2, 1, 6
101, 27, 20, 7, 6, 1, or $\frac{27}{101}$

5th " 3, 1, 2, 1, 6, 2
217, 58, 43, 15, 13, 2, 1, or $\frac{58}{217}$

and so on for any subsequent approximation.

If the above division had been carried on to the last divisor, or unity, as the numbers are prime to each other, the original fraction would have been obtained again, thus :

3	1	2	1	6	2	50	1	5	1	1	2	
367459	98215	72814	25401	22012	3389	1678	33	28	5	3	2	1

the upper line being the successive quotients, and the lower the continued products, with the addition of the preceding number, (or last numerators.)

Suppose the following fraction, (to give one more example,) which represents approximate numbers expressing the diameter and the circumference of a circle,

$$\frac{100000000}{314159265}$$

The division gives as follows :

```
100000000)314159265(3
                14159265)100000000(7
                              885145)14159265(15
                                      5307815
                                       882090)885145(1
        3055)882090(288
             27169
              26690
               2250)3055(1
                    805)2250(2
                         640)805(1
                             165    &c.
```

1st approximation. 3 , 1 or $\frac{1}{3}$

2nd. " $\left\{\begin{array}{c} 3 , 7 \\ 22 , 7, 1 \end{array}\right.$ or $\frac{7}{22}$

3d. " $\left\{\begin{array}{c} 3 , 7 , 15 \\ 333 , 106 , 15, 1 \end{array}\right.$ or $\frac{106}{333}$

4th. " $\left\{\begin{array}{c} 3 , 7 , 15 , 1 \\ 355 , 113 , 16 , 1, 1 \end{array}\right.$ or $\frac{113}{355}$

5th approximation.

$\left\{\begin{array}{c} 3 , 7 , 15 , 1 , 288 \\ 102573 , 32650 , 4623 , 289 , 288 , 1 \end{array}\right.$ or $\frac{32650}{102573}$ &c.

Examples.

1st.	$\frac{794973}{1674210}$;	5th.	$\frac{692}{1817}$;	9th.	$\frac{1015}{1827}$
2nd.	$\frac{38125}{516412}$;	6th.	$\frac{888}{2777}$;	10th.	$\frac{1879}{2425}$
3d.	$\frac{5967}{13843}$	7th.	$\frac{1775}{1823}$	11th.	$\frac{7714}{52763}$
4th.	$\frac{81097}{649321}$	8th.	$\frac{337}{759}$	12th.	$\frac{9864}{37652}$

CHAPTER V.

Of Decimal Fractions.

§ 51. In explaining the decimal system of our usual arithmetic, we have seen: that every figure designates a quantity ten times greater, when it stands one place farther to the left hand from the unit, than when in the place preceding it, and therefore conversely, the figure in the next place to the right hand is ten times smaller than the same figure in the next place to the left of it.

If we continue this reasoning below the unit of whole numbers, and after having marked that place by a (,) and give denominations to the parts of unit, according to the same system, we shall get successively for the resulting places, the denominations of tenth parts, (or $\frac{1}{10}$,) hundreth parts, (or $\frac{1}{100}$,) thousandth parts, (or $\frac{1}{1000}$,) and so on, to any part or subdivision, however minute, of the decimal system; as for instance, 3,45672 would be 3 units, 4 tenths, (or $\frac{4}{10}$,) 5 hundreths, ($\frac{5}{100}$,) 6 thousandths, $\frac{6}{1000}$,) 7 ten thousandths, ($\frac{7}{10000}$,) 2 hundred thousandths, ($\frac{2}{100000}$,) or the whole would be,

$$3 + \frac{4}{10} + \frac{5}{100} + \frac{6}{1000} + \frac{7}{10,000} + \frac{2}{100,000}$$

where it is evident that the writing of the denominators can be spared, because the successive diminution of value of the places is known by the system; therefore the usual, and easiest, way of reading these fractions is, after having mentioned the whole numbers, to mention the (,) and read the subsequent numbers simply as they occur, leaving the denominations out, as understood from the principles of the system. We have hence an easy mode of expressing any fraction in the same system as the whole numbers, either in full,or by approximations to any desirable extent, therefore minuteness of accuracy.

§ 52. We have already seen that proper vulgar fractions result from a remainder of a division smaller than the divisor, as a mere expression of the unexecuted, or, by the means hitherto explained, unexecutable division; by decimal fractions we are on the contrary, enabled to express these quantities, by continuing the division, according to the same law that is used in the system of numbers; and all the difference between operations with these quantities, and those with whole numbers, will evidently consist in pointing out the place where the whole numbers end, and these fractions begin; all the rules which will be found hereafter, for the mechanical execution of the four rules of arithmetic in decimal fractions, will therefore merely relate to the determination of the place of the decimal mark.

To continue the division that is required to obtain the decimal fraction, after the number in the dividend has become smaller than the divisor, it becomes necessary, to reduce the remainder to the same kind of unit which will follow in the quotient, and as this will be ten times smaller, the dividend will represent in it a ten times larger number; to do this we have only to multiply it by ten, which is done by the simple addition of a (0) on its right hand side; this will enable us to continue the division; as this takes place at every step, it is only required to repeat it also at every step, as in the following example:

Let the division be continued below the unit:

$$\frac{34721}{763} = 45{,}5058977, \&c.$$

4201
3860
4500
6850
7460
5930
5890
549, &c.

Here, after having obtained 45 as a whole number, instead of expressing the fraction $\frac{386}{763}$ as a vulgar fraction, an 0 has been added to the remainder 386, which presents 3860, and is again divisible by the divisor 763; at this place therefore, or after 45 in the quotient, the distinctive mark or (,) was placed, and the division continued, as in common numbers, with the constant addition of a 0, to every remainder, to make the continued division possible; at the remainder 45, the addition of one 0, making a dividend, 450, still smaller than the divisor, the quotient was 0, as in any other division, and the addition of a second 0, making 4500 in the dividend, gave the quotient 5, proceeding in all such cases as has been shown in common division; the division, which here stops at the seventh place of decimals, might perhaps continue in this case without ever closing.

§ 53. By the same principle we can express any vulgar fraction in decimal fractions, either terminated, or continued as far as desired, or solve the

PROBLEM. *To reduce vulgar fractions into decimal fractions.*

Put an 0, *in the place of the unit in the quotient, with a* (,) *after it, and an* 0, *at the right hand side of the numerator, divide as in common division, adding to every remainder an* 0, *and continuing this division as far as desired, or until recurring numbers occur.*

Example. $\frac{3,0}{7} = 0,42857142$

20
60
40
50
10
30
20
6 &c.

Placing the 0 in the quotient, and 0 in the numerator, the division is continued as in common numbers, with the constant addition of a (0) to the remainder, until we again meet the same quotient, 42, and remainders 2 and 6, which had been obtained at first; which are called recurring numbers, and indicates that the same series of numbers would repeat themselves; and which may therefore be done, as far as required, without calculation. Such series of recurring numbers are called circulating decimals.

Example: reduce $\frac{1}{3} = 0,3333$

10
10

Here evidently we constantly obtain from the very beginning, the same quotients and remainders; the calculation need therefore, not be continued. Of this kind are a number of other fractions, called repeating decimals.

In the following example, we obtain a complete expression in decimals,

$\frac{1,0}{8} = 0,125$; which terminates the division of itself.

20

40

Examples. Reduce;

$\frac{1}{8}$; $\frac{7}{8}$; $\frac{9}{15}$; $\frac{4}{5}$; $\frac{2}{3}$; $\frac{1}{5}$; $\frac{1}{6}$; $\frac{1}{7}$;

$\frac{1}{4}$; $\frac{3}{16}$; $\frac{9}{11}$; $\frac{3}{4}$; $\frac{6}{7}$; $\frac{2}{5}$; $\frac{3}{8}$; $\frac{7}{9}$;

§ 54. ADDITION OF DECIMAL FRACTIONS.

By the principles explained in § 51, it is evident, that all operations upon decimal fractions must proceed from the unit to the lower numbers, or subdivisions, according to the decimal system, below the unit, by the same law as above it, and the *Principles* applied to whole numbers apply equally to *Decimal fractions*; the numbers being all of the same system, the carrying of the individual sums will be the same as in whole numbers, and each kind of number will thereby stand in its proper place; therefore, also, the decimal mark will not change its position. If any one of the given numbers should have no whole numbers, in order to avoid all ambiguity, the place of the unit must be filled with a 0 followed by a (,). Hence derives the following Rule.

Place the numbers under each other, in such a manner: that the units may stand under the units, and all the numbers, at equal distances, to the right, or to the left of the units, may fall under each other; then add them as in common numbers, beginning at the right hand figure, and place the decimal mark in the result, under the decimal mark of the given numbers.

Example. To make the sum, or execute:
3,4612 + 21,34891

3,4612
21,34891

24,81011

The following is an example where the decimals go farther in the smaller number than in the larger:

Example of several numbers;
13,76094
0,3809673
142,012
0,39052

156,5444273

Examples.

1st. 3,09047+0,097384
2nd. 7,914+31,96147+0,0316+0,000754.
3d. 9,4183+0,9473101+0,9070407+6,011
4th. 0,0076+0,0106+0,000049+0,000039
5th. 0,9047+100,1047+19,735+72,7641
6th. 0,0409+73,9043+2,765+26,30107

§ 55. SUBTRACTION OF DECIMALS.

The principles stated above of course lead also in this case; the numbers being written under each other so as to make the unit figure the leading figure, each kind of decimal will stand under its corresponding one, and the subtraction will be in no wise different from that of whole numbers; observing: that when no lower decimals are in the larger number, from which the subtraction is to be made, (these being not written,) they are of course supposed 0's, and the subtraction is made by means of borrowing to the left, in the same manner as if in whole numbers there were 0's, in the lower places of figures. This leads therefore to the following:

Rule. *Place the numbers under each other as in addition,* (the smaller being usually written below the larger) *and subtract as in common subtraction, beginning at the right hand figure, and place the decimal mark in the difference under the decimal mark of the two given numbers.*

Example. 3,490864 — 2,74962, write thus:

3,490864
2,74962

0,741244

3,490864

There being no number to subtract from the first 4, it is unchanged in the remainder, and the other numbers follow, as in common subtraction.

Example. 0,9832
0,4986735

0,4845265

3,9832000

Here for the subtraction of the first number or 5, the supposed lending from before has given 10, and the remainder became 5; then the next lending being again from before, but one being borrowed from the 10, leaves only 9; the 3 is subtracted from 9, remainder 6; in like manner in the next we obtained 9 — 7 = 2; and then the first number above, namely the 2, appears as diminished by 1, and the next subtraction is made from 11, by borrowing again as in whole numbers: the rest is evidently as in whole numbers.

Examples. 1*st.* 13,409 — 7,946
2*nd.* 0,9047 — 0,83946
3*d.* 0,09109 — 0,009867
4*th.* 3,4981 — 2,95673
5*th.* 0,994098 — 0,84376
6*th.* 10,843 — 7,3942
7*th.* 0,0194 — 0,008643
8*th.* 9,094 — 8,3946
9*th.* 8,947 — 3,60409

§ 56. MULTIPLICATION OF DECIMAL FRACTIONS. From the principles of Decimal Fractions it is evident; that their multiplication can, in itself, have no other rule than that of whole numbers. In respect to the value of the resulting decimals it is easy to observe, that as they are fractions with the denominator 10, each of its preceding number; they keep this quality in the result, and therefore present always in their product the product of these quotients, which, though not expressed by a denominator, is indicated by the place assigned to the decimal mark. Therefore, as we have seen before, units multiplied by units give in product units, or tens and units, so in decimal fractions *tenths* multiplied by tenths will give *hundreds* and *tenths* or *only hundreds*, and thus for every other denomination. This principle evidently causes the decimal mark to recede one place for every time that it is multiplied by a decimal, and in the multiplication of decimals, distant from each other, as many places as both decimals together indicate; because, the product of any tenths, hundredths, thousandths, &c., into any other tenths, hundredths, thousandths, &c., will be the unit followed by as many 0's as are contained in the supposed denominators of the two fractions together.

This proves therefore, for multiplying decimal fractions, the following

RULE. *Multiply the factors together, as in whole numbers, and cut off as many places of decimals, from the right hand side towards the left, as there are decimals in both the factors, place there the decimal mark.*

To execute this multiplication, it is best to begin by the unit, and multiply by the other numbers to the left, and then to the right, advancing the result in the first case to the left, and receding with the results of the decimals towards the right side, in the second case; in that order in which the multiplication with whole numbers in the same ranks would indicate; the first result, that is the result of the unit, will by this process, at once determine the place of the decimal.

Execute the following multiplication. 3,476 × 5,82;

thus: 3,476
5,82

17,380
2,7808
6952

20,23032

Here the multiplication by 5, as units, has given to every number in the multiplication the same place as it occupied before; the following numbers having receded towards the right, according to the rank they naturally have already, occupied their proper place, and we have obtained in the result 5 places of decimals, exactly as many as both factors have together, and as the rule above prescribes; (for which therefore it might form a practical deduction or proof.)

Let the following examples be executed, to illustrate further the practical application and its consequences.

36,452	172,7892	0,5378
0,4937	56,32	0,0624
14,5808	1036,7352	0,032668
3,28068	8639,460	10756
109356	51,83674	21512
255164	3,455784	0,03355872
17,9963524	9731,488624	

The first example above gives a result that makes the product recede one place in the decimals, because the first multiplier is a decimal, this determining the decimal place, all the others follow by themselves, the result receding always one place, and ultimately giving as many places of decimals as there are in both the factors taken together. When in the second example the multiplication by 6, as unit, was performed, each number resulting kept the place it occupied in the multiplicand, the decimal place being determined by this. The next multiplication with 5, or properly 50, gives the advance towards the left, according to common multiplication of whole numbers, the two decimals being made to recede each one place farther to the right, keep the final result in its proper order, the number of decimals are thus determined by the mere addition, and are conformable to the above rule.

In the third example: the $\frac{6}{100}$ into the $\frac{5}{10}$; gives evidently, $\frac{30}{1000}$: which assigns to this result the place seen in the example; the others follow by themselves; but if we had not attended to this, the rule above would give the coincidence with the result obtained, as may be easily seen.

Examples. *1st.* 3,415 × 0,7604
2nd. 17,4089 × 0,03964
3d. 2,3918 × 321,6402
4th. 0,00964 × 0,03458
5th. 0,90807 × 0,00652
6th. 372,0496 × 0,000985
7th. 0,008643 × 0,00004329

§ 57. It is most generally needless to calculate to the full extent of all the decimals of both factors, the one or the other factor commonly indicates the number of decimals to which it is intended to carry the accuracy; and we may dispense with the smaller decimals, so much the rather, as they will at all events not be absolutely accurate, if the decimal fractions are not themselves exact; in consequence of the absence of the smaller decimals, that would influence the places taken beyond the lowest that is given in one or the other factor; for this reason it is desirable to have an easy, and exact mode of executing this abridged multiplication, which is as follows:

Multiply by the greatest number of the multiplier first, and determine the place of the decimal; (as in the preceding rule;) *then mark this number, and also the lowest decimal of the multiplicand; then take the next lower number of the multiplier, and multiply all the multiplicand by it, taking from the product of the decimal marked off, only the part which is to be carried forward to the next place, using the ten nearest the result either above or below; write the product under the foregoing, so that the first figure to the right comes under the first figure of the foregoing product; thus continue as long as there are figures in the multiplier, always marking off one figure in the multiplicand for each factor of the multiplier, and making the addition of the carrying as before; the decimal mark will* *take its place according to the determination of the first number used as a multiplier.*

Example. The first of those given above, executed according to this rule, will show that in this manner regularity is insured, and the deviation from the full result obtained above will not extend farther than the last place of figures, or rather only the next after it, as it does not differ a whole unit of that place.

$$\begin{array}{r} 3\dot{6},\dot{4}\dot{5}\dot{2} \\ 0,\dot{4}\dot{9}\dot{3}\dot{7} \\ \hline 14,5808 \\ 3,2807 \\ 1094 \\ 255 \\ \hline 17,9964 \end{array}$$

The first line is the product of $\frac{4}{10}$ into the multiplicand, thence its product by the unit must give tenths, which determines all the rest.

In the second line we had to carry for the product of 9 × 2 = 18, which being nearer to 20 than 10, the tens to be carried were two; so that when we afterwards made 9 × 5 = 45, we added 2, making 47, the 7, being placed, the 4, carried, and the operations continued as in common numbers; the rest of the operation is exactly the same with respect to the remaining numbers of the multiplicand; for the third line we had 3 × 4 = 12, and the 3 × 52 from before, giving 2 to carry, rather than only 1, we added 2, that is, we made 12 + 2 = 14, then continued 3 × 6 = 18, and 1 carried, = 19, and so on; the addition is as before.

Examples. 1*st.* 0,4965 × 9,0829
2*nd* 7,6048 × 3,97652
3*d.* 0,0935 × 10,4342
4*th.* 0,004093 × 21,35647

5th. 0,0986 × 0,004732

6th. 0,00947 × 0,0009375

7th. 9,07093 × 0,098643

8th. 0,0004319 × 1097,6245

9th. 194,3062 × 0,0084837

10th. 3461,279 × 0,0008765

11th. 54,3967 × 0,007583

12th. 315,0463 × 0,000497

§ 58. DIVISION OF DECIMAL FRACTIONS.

From the manner the origin of Decimal Fractions has been deduced, it has already been seen: that the decimals began whenever the divisor was larger than the dividend, or which is the same thing, when an 0 became necessary to be added; in other words, when it became necessary to recede towards the right farther than the quantities of the dividend furnished numbers of the value of those of the divisor; as many such steps therefore, as it may become necessary to make until a figure, actually significant, can be obtained in the quotient, as many 0's will precede it, the 0 of the unity place being counted as the first; or the place of the first significant decimal will be that indicated by the number of steps it was necessary to recede.

It will be most easy in this place, and will furnish us with the clearest method of accounting for the necessary steps, to proceed from the relation of the unit in the dividend and the divisor, as a leading principle for the determination of the decimal mark; and in cases where no whole number is obtained in the quotient, it will be entirely referable to common division continued to decimals, as employed by us in elucidating the principle of decimal fractions. The rule resulting will therefore be this:

Begin the division as in whole numbers, and place the first decimal resulting, as many places to the right of the decimal mark, as it has been necessary to recede from the first figure in the divisor, to obtain in the dividend a number sufficiently large to be divided by the divisor.

Example. Divide $\frac{3,46921}{4,327} = 0,8017587 + \&c$

7610
32830
25410
37750
31340
1051 + &c.

Here it is evident, that 4 not being contained in 3, whatever may be the following decimals, the divisor cannot be contained in the dividend a whole number of times, thence there is no whole number in the quotient, therefore a 0 is written in the place of the unit; the next receding in the dividend giving a significant figure; this falls in the first place of decimals. And now the division being performed, as in whole numbers, and the 0 added always, after there were no numbers more to be taken down from the dividend, the division can be carried as far as may be desired, for the decimals determine themselves without any further care. But it is evident here again: that the division to greater extent than is warranted by the numbers given, will not give the full accuracy, when the decimals are not determined ones, and only approximations; for the 0's set down should evidently always be the figures which would have followed in the dividend; and the products to be subtracted, should be affected by the lower decimals, which are missing in the divisor.

The two Examples that follow, will show more of this application.

$\frac{4,2769}{567,432} = 0,007537 + \&c.$

3048760
2116000
4137040
165016

$\frac{0,00042769}{0,0567432} = 0,007537 + \&c.$

3048760
2116000
4137040
165016

These examples, showing both cases of whole numbers and decimals, with a denominator exceeding the numerator, are both equal applications of the rule found above; and their result is the same; as the one is evidently the product of the other, by an equal multiple of 10, in numerator and denominator; in both cases it was necessary to recede three steps, to obtain a significant number in the quotient; therefore, the first number in the quotient is in the third place of decimals; or, we may say, it was three times impossible to divide; therefore we have three 0's the 0 of the unit place being counted, as of course, because the first time the division was not possible, was that of whole units.

In the same way as we have found in vulgar fractions: that one fraction may be contained in another fraction a whole number of times, as well as one whole number in another; so, of course this also takes place in decimal fractions, as in the two following examples, which again present the same quotient.

$\frac{452,96738}{15,3129}$ = 29,580770+ &c.	$\frac{0,045296738}{0,00153129}$ = 29,580770+ &c
146 7093	1467093
8 89328	889328
1 236830	1236830
117980	117980
1078930	1078930
70670+ &c.	70670+ &c.

These examples first admitted of division by whole numbers, because 15 is evidently contained in 452 a whole number of times, which we found to be 29: then the decimals began; these two whole numbers are evidently easily determined, as the divisor has only two places of figures in the whole numbers of the first example, and the dividend 3. In the corresponding places of the second example 1 thousandth admits evidently also twice the product of whole numbers in the 45 thousandth, and gives therefore equally a quotient of tens and units; so that both are already twice divisible, before recourse is had to the decimals from the dividend, for the subtraction of the products of the whole numbers of the divisor into the quotient.

Remark. If it be required to perform any one of the four rules of arithmetic, between decimal and vulgar fractions, the decimal fractions are to be considered as whole numbers, paying, of course, due attention to the place of the decimal mark; this will therefore need no special explanation. But it will in many cases, be most convenient to reduce the vulgar fraction into a decimal fraction, and then proceed upon the principles of decimal fractions. This being therefore a subject depending on the judgment of the calculator, the principles of which have been explained sufficiently, it will not need here to be treated separately.

Examples.

1st. $\frac{7,431}{1349,638}$; *2nd.* $\frac{0,0312}{7,469}$; *3d.* $\frac{0,009662}{0,08735}$;

4th. $\frac{0,964}{92,738}$; *5th.* $\frac{3,4829}{0,09634}$; *6th.* $\frac{6,00964}{3,0975}$;

7th. $\frac{34,798}{0,8109}$; *8th.* $\frac{94,3218}{5,4396}$; *9th.* $\frac{0,0000768}{9,0973}$;

10th. $\frac{0,003497}{0,00975}$; *11th.* $\frac{973,793}{0,96094}$; *12th.* $\frac{0,000473}{9,0865}$;

CHAPTER VI.

Of Denominate Fractions.

§ 59. I take the liberty of calling all those subdivisions of an accepted unit that have received particular names, and which properly form fractions of this unit, with a certain conventional denominator, that is therefore always understood: *Denominate Fractions*; such are all the subdivisions of measures of any kind; of length, surface, or solidity, weights, money, time, &c.

In order to make use of these fractions in arithmetic, it is necessary to know their conventional denominators, or to be able to say how many units of each subdivision make a whole, or a unit of a higher subdivision; as for instance, the general division of pound (money) into shillings, pence, and farthings; where the general habit is that 1 shilling = $\frac{1}{20}$ of a pound; 1 denier, or penny = $\frac{1}{12}$ of a shilling; 1 farthing = $\frac{1}{4}$ of a penny, Or, in the other manner of expressing it,

$$£1 = 20s;\ 1s = 12d;\ 1d = 4f.$$

Of these subdivisions, old habits, unconnected in their origin, and therefore devoid of system, have introduced such a variety, that it is necessary to have tables in order to recall them to memory; such tables will be placed at the end of this book, to which I shall add the approximate or full decimal expression of the unit of each subdivision in the other, and in the whole, as it is evidently possible to express them all in decimal fractions, either exactly or approximately. Certain signs have been given to all these subdivisions, to abridge their notation; these will be learnt from the tables.

It is usual also in writing these kind of quantities to write always the smaller ones to the right hand side, and progress towards the left, in proportion as the quantities indicated become larger.

In thus stepping aside, from the simple theory of a system, to a mere practical habit, we shall soon feel what an advantage it would be in all transactions where quantity is concerned, to have a regular and unique system for them all; but the attempt, so often made, has always been frustrated, by the unwillingness of men engaged only in their private concerns, to all mental motion or exercise, not directly advancing their private aims. Similar systems had been in use in common arithmetic, before the adoption of the decimal system of numeration, the advantages of which soon expelled them from theoretic arithmetic.

It is evident that the difficulties to be vanquished in this part of arithmetic, consist only in the attention that is required to be paid to the effect of the irregular system of subdivision, which determines the principles of what may be called *carrying* from one denomination to the other; the rules discovered hereafter, therefore, chiefly refer to this operation; they will not need any proof, as they have only the arbitrary subdivisions for their principle; and for their aim, to facilitate the several processes. They will therefore be given simply, with a few examples for illustration; their proof, as far as arithmetical principles are concerned, lying always in the principles of calculation already explained; and their combination will be reserved for the practical part of this treatise.

§ 60. ADDITION OF DENOMINATE FRACTIONS. Rule. *Write the numbers of each denomination under each other, distinguishing them by points; add them as whole numbers, beginning at the most right hand figures, and carry from one denomination to the other, according to the value of the subdivision, in parts of the next superior quantity, that is, dividing the sum obtained by the denominator of the fraction, indicated by this subdivision.*

Example—in feet, inches, and tenths of inches, 12 *in.* = 1 *f.* affords the principle of carrying from inches into feet; the mode of carrying the tenths of inches being as explained in decimal arithmetic.

Find the value of 12 *f.* 7,6 *in.* + 3 *f.* 4,9 *in.* + 2 *f.* 11,2 *in.*

f.	*in.*
12.	7, 6
3.	4, 9
2.	11, 2
18.	11, 7

Here the sum of inches being 23; 12 are taken away, to carry as one unit to the feet; there remains 11.7 inches; the rest is exactly like the addition of whole numbers.

Example in weight, of pounds Troy; the subdivisions of which are,

1 *lb.* = 12 *oz*; 1 *oz.* = 20 *dwt*; 1 *dwt.* = 24 *gr.*

	lb.	*dwt.*	*oz.*	*gr.*
To add	7.	10.	14.	12
	19.	6.	17.	14
	6.	11.	15.	19
	34.	5.	7.	21

Adding the first column to the right, or of grains, what is over 24 gives 21, to set under this denomination, and 1 dwt. is carried to the next denomination, or dwt. column. This second column being then added, gives 47 dwt. = 2 oz. + 7 dwt; therefore the 7 dwt. are placed under that column, and 2 oz. are carried to the ounces; the ounces added, with the carrying, give 29 oz. = 2 lb. + 5 oz; the later placed under ounces and the 2 pound carried give ultimately, by the addition of the last column the number of whole units of pounds, 34; then the whole sum is expressed.

EXAMPLES.

Apothecaries' Weight.

lb.	*oz.*	*dr.*	*scr.*	*gr.*
13.	7.	5.	1.	15
5.	3.	1.	9.	10
2.	10.	7.	1.	16
4.	9.	3.	2.	18
22.	11.	6.	0.	12

Wine Measure.

ti.	*gal.*	*qt.*	*pt.*
7.	29.	3.	1
18.	40.	1.	0
3.	35,	0.	1
12.	14.	2.	0
9.	41.	3.	1

Long Measure.

m.	*fur.*	*p.*	*f.*	*in.*
6.	4.	12.	7.	9
3.	7.	14.	12.	5
1.	3.	0.	1.	10
9.	5.	35.	14.	7
2.	6.	29.	15.	3

Square Measure.

a.	*r.*	*p.*	*f.*
7.	2.	29.	123
9.	3.	16.	99
5.	1.	35.	201
3.	0.	14.	69
6.	2.	12.	87

Cubic Measure.

fath.	*yd.*	*f.*	*in.*
8.	7.	23.	216
9.	2.	14.	621
5.	5.	22.	907
7.	8.	15.	681
8.	2.	11.	809

Cloth Measure.

yd.	*q.*	*nl.*	*in.*
17.	2.	3.	1,5
75.	3.	1.	2,0
39.	1	2.	1,7
56.	3.	3.	2,1
21.	2.	1.	1,0

Avoirdupois Weight.

ton.	*cwt.*	*q.*	*lb.*	*oz.*
11.	12.	2.	21.	13
3.	15.	1.	12.	7
2.	18.	3.	7.	9
6.	10.	2.	24.	11
5.	7.	0.	19.	14

Dry Measure.

bush.	*pk.*	*g.*	*pt.*
17.	3.	1.	7
22.	1.	1.	6
13.	2.	0.	4
7.	1.	1.	5

§ 61. SUBTRACTION OF DENOMINATE FRACTIONS. RULE. *Write the denominations of the subdivisions of the quantity to be subtracted, under the same denominations in the quantity whence they are to be subtracted, and subtract in each column the lower from the upper, beginning at the lowest denomination; and in borrowing from a superior denomination, give to the unit borrowed the value it has in the lower subdivision in which it is used.*

This rule is evident from the simple inversion of what is directed to be done in addition, and is analogous to the rule in the decimal system, as is evident. These subtractions evidently admit of proof, by the addition of the remainder to the number subtracted, as is the case in all subtractions.

Example in feet, inches, and tenths.

f.	*in.*
17.	7, 3
8.	9, 6
8.	9, 7
17.	7, 3

Example in pounds Troy.

lb.	*oz.*	*dwt.*	*gr.*
22.	7.	6.	5
14.	9.	16.	12
7.	9.	9.	17
22.	7.	6.	5

The borrowing in the first example, from the feet to the inches, has given 12 + 6 in. = 18 in. from which 9 taken left 9, the decimals having been treated as usual; then 16 — 8 ft. = 8 ft. gave the remainder 8, of the whole feet. So also in the second example, we had first, by borrowing to the grains 24 + 5 — 12 = 17 (grains;) then in the second column 20 + 5 — 16 = 9 pennyweights, the 6 having been reduced to 5 by the preceding borrowing; in the third column 12 + 6 — 9 = 9 ounces, and lastly 21 — 14 = 7 pounds, the borrowing having been made throughout according to the dictates of the arbitrary subdivision, and the numbers from which the units were successively borrowed, having always been diminished by a unit.

EXAMPLES.

Troy weight.
17 *lb.* 11 *oz.* 10 *dwt.* 21 *gr.* — 8 *lb.* 10 *oz.* 19 *dwt.* 22 *gr.* =
8 *lb.* — 6 *lb.* 6 *oz.* 17 *dwt.* 13 *gr.* =

Apothecaries' weight.
9 *lb.* 0 *oz.* 3 *dwt.* 1 *scr.* 12 *gr.* — 6 *lb.* 8 *oz.* 6 *dwt.* 2 *scr.* 11 *gr.* =
13 *lb.* — 9 *oz.* 6 *dwt.* 2 *scr.* 15 *gr.* =

Long Measure.
7 *poles* 3 *yd.* 2 *ft.* 9 *in.* — 3 *poles* 4 *yd.* 2 *ft.* 19 *in.* =

Square Measure.
7 *acres* 1 *rood* 6 *poles* 6 *ft.* — 5 *acres* 3 *roods* 25 *poles* 8 *ft.* =

Cloth Measure.
17 *yd.* 2 *qr.* 1 *nl.* 1 *in.* — 15 *yd.* 3 *qr.* 3 *nl.* 2 *in.*

§ 62. MULTIPLICATION OF DENOMINATE FRACTIONS. From the two different ways in which it has been seen that denominate fractions may be compared, namely, as units, of which a certain number form another unit, or as fractions of the preceding, or rather the highest unit, with a certain denominator understood, it may be inferred: that the multiplication of them can be executed in two different ways.

The first, *using the given numbers as units,* will form fractions, of a denomination adapted to the conventional system of subdivision, which mode is exactly analogous to what has been done in decimal fractions. This is often called *Cross Multiplication.*

The second consists in using the lower units that are given *as fractional parts of the whole,* and takes the products of the multiplier into the multiplicand as such, distributed in parts that are best adapted to the easy division of the multiplicand, which are thence often called *aliquot parts,* and expresses the results in the same unit, and its subdivisions; the final result is obtained by the addition of the products; this method is usually called *Practice.*

In the application of multiplication to this species of quantities, we are limited by their nature to those which are capable of producing things really existing in nature; as lineal measures into each other, which produce surfaces, and these again into lineal measures, which produce solids.

But such quantities as money into money, weight into weight, being incapable of producing any possible result, cannot be subject to multiplication as such.

The multiplication of denominate fractions of different kinds into each other, as for instance, money into weight or measure, &c. is to be done by the second method, or *Practice*, or by reduction to any one *determined unit*, and its fractions, if any are in it; then the ground of calculation is determined by their conventional relative value. They are in fact only calculable as certain *fractions* of the unit. This being entirely practical, will be considered only in the second, or applied part.

§ 63. To multiply by the first method, or *Cross Multiplication*, we have the following rule, grounded upon the general principles of fractions.

Multiply all the columns of the multiplicand successively by all the numbers of the columns of the multiplier, beginning at the right hand figure, and carry, in each passage to a higher column, according to the value of the subdivisions made use of; place the results of equal quantities under each other; their sum in the result will give the whole quantities and the subdivisions, according to the same scale as the preceding subdivisions; that is, the denominators becoming equally products of the denominators indicated by the subdivisions.

Example. To execute 7 ft. 2 in. × 6 ft. 5 in.

ft.	*in.*	
7.	2	
6.	5	
2.	11.	10
43.	0	
45.	11.	10

By multiplying here 2 in. into 5 in. we have properly made $\frac{2}{12} \times \frac{5}{12} = \frac{10}{144}$; therefore we have obtained a subdivision of the unit one degree lower, and upon the same scale, than those employed; as in decimal fractions; therefore, also in writing the result, this has been removed one step more to the right. In making

$$7\,ft. \times 5\,in. = 7 \times \frac{5}{12}\,ft. = \frac{35}{12} = 2 + \frac{11}{12},$$

we have obtained first twelfths, and then, by reduction to whole numbers, 2 whole quantities, and $\frac{11}{12}$ of a foot; the result of 2 *in.* × 6 *ft.* $= 6 \times \frac{2}{12} = \frac{12}{12} = 1$, has given by the same principles, one foot to carry to the next result; then, by the multiple of the feet into the feet, with this addition, we have 6 × 7 + 1 = 43 ft; the final sum is obtained as in addition, but presents an inferior subdivision, one degree lower in the same scale of subdivision that is used, or twelfths into twelfths (which are called *lines*, where this subdivision is used;) thence the above result is $= 45 + \frac{11}{12} + \frac{10}{144}$. or 43 square feet 142 square inches.

If we multiply again (for example the above result) by a lineal dimension, we shall obtain a solid, expressed in the same system of subdivision; thus:

$$\begin{array}{rrrr}
ft. & in. & l. & \\
45. & 11. & 10 & \\
5. & 7 & & \\
\hline
26. & 9. & 10. & 10 \\
229. & 11. & 2 & \\
\hline
256. & 9. & 0. & 10
\end{array}$$

Here we again obtain, by the very same process as above, and for the same reason, a denomination still lower in the same scale of subdivision,

$$\text{or} = 256 + \frac{9}{12} + \frac{10}{12 \times 12 \times 12} = \begin{array}{l}256 \text{ cubic feet}\\ 1306 \text{ cubic inches.}\end{array}$$

§ 64. *Multiplication of Denominate Fractions as fractional parts of the whole, or Practice.* The nature of this operation, as stated above, evidently leads to the rule,

Multiply the whole and fractional parts of the multiplicand by the whole numbers of the multiplier; then distribute the fractional parts of the multiplier into such as are most easily taken; take such parts of the whole and fractional part of the multiplicand as will be indicated by them, and add all these parts for the final result.

By this operation the products of the different fractions, distributed for the convenience of the operation, being partially taken, the proof of the rule lies in the simple multiplication of fractions, and this is only repeated; it is generally convenient to divide the fractions of the multiplier so that the smaller subsequent parts are again fractions of the first.

For the purpose of comparison with the other mode, we shall use the example already given.

$$\begin{array}{rl}
\textit{ft.} & \textit{in.} \\
7. & 2 \\
6. & 5 \\
\hline
43. & 0 \\
2. & 4\frac{2}{3} = \frac{1}{3} = 4 \textit{ in.} \\
0. & 7\frac{1}{6} = \frac{1}{4} \times \frac{1}{3} = 1 \textit{ in.} \\
\hline
45. & 11\frac{5}{6}
\end{array}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} 5''$$

Here, after multiplying the 7 ft. 2 in. by 6, the whole number, giving 12 in. + 42 ft. = 43 ft. the 5 in. of the multiplier are divided into 4 in. = $\frac{1}{3}$ ft. and 1 in. = $\frac{1}{4}$ of 4, in. or $\frac{1}{4} \times \frac{1}{3}$ ft. the $\frac{1}{3} \times 7$ ft. giving 2 ft. and 1 ft. remaining, which gives 12 in. to add to the 2 in., the $\frac{1}{3}$ of the (12 + 2) in. = 14 in. being taken, gives $4\frac{2}{3}$ in., as in the second line; the second fractional part being $\frac{1}{4}$ of that, or of 2 *ft.* + 4 $\frac{2}{3}$ *in.* = 28$\frac{2}{3}$ *in.*, the $\frac{1}{4}$ of which is 7 + $\frac{2}{3 \times 4}$ = 7 + $\frac{1}{6}$ in., gives the third line; the addition of the products is easily understood from the addition of fractions, as $\frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \frac{5}{6}$, and the rest is like the addition of denominate fractions.

Let this example be continued, as was done in the former case.

$$\begin{array}{rl}
\textit{ft.} & \textit{tw.} \\
45. & 11\frac{5}{6} \\
5. & 7 \\
\hline
229. & 11\frac{1}{6} \\
22. & 11\frac{11}{12} = \frac{1}{2} = 6 \text{ in.} \\
3. & 9\frac{71}{72} = \frac{1}{6} \times \frac{1}{2} = 1 \text{ in.} \\
\hline
256. & 9\frac{65}{72}
\end{array}
\left.\begin{array}{l} \\ \\ \end{array}\right\} 7 \text{ in.}$$

We have 5 × $\frac{5}{6} = \frac{35}{6} = 4 + \frac{1}{6}$, for the product of the whole number 5 into the fraction; then 5 × 11 + 4 = 59 in. = 4 ft. 11 in., from the whole into the inches, with the carrying, the rest then as whole numbers, or 45 × 5 + 4 feet. For the 7 in. we take 6 in. = $\frac{1}{2}$ ft., and 1 in. = $\frac{1}{6} \times \frac{1}{2}$ ft.; therefore $\frac{1}{2}$ × (45 ft. 11$\frac{5}{6}$ tw.) giving the second line easily; and the third line being $\frac{1}{6}$ of that, presents $\frac{22}{6}$ ft. = 3 ft. + $\frac{48}{6}$ tw; the second part or $\frac{48 \text{ in.} + 11 \text{ tw.}}{6}$ = 9 in + $\frac{5}{6}$; and the fractional part of this, making in $\frac{1}{12}$ as the upper fraction is $\frac{10}{12}$, which added to $\frac{11}{6 \times 12}$, gives $\frac{71}{12 \times 6} = \frac{71}{72}$.
The addition is executed in the fractional part by reducing them all to the denominator 72, for their common denominator; the whole twelfths are then carried to the next addition, which is executed as shown in its place.

If any such denominate fraction of any kind is to be multiplied by a whole number, it is evident, that nothing is required but to multiply each of the parts by this number successively, in the above order; carrying according to the principle of the given arbitrary subdivision, exactly as was done with the first, or whole number above. This is evidently admissible with any subdivision, and needs no separate explanation, as it has actually been already given.

§ 65. The two preceding modes of performing the multiplication of denominate fractions being evidently cumbersome when applied to great calculations, and when the fractional parts, or lower denominations, are not easy aliquot parts of the whole, it will be often most convenient, te reduce the factors to whole and decimal fractions, by the methods taught in their proper place, and then to proceed by multiplication of decimals; for this purpose the fractions would be marked with their divisors, according to the habitual subdivision, and the execution of the division will give the proper decimal.

Example. Let the preceding question be executed in this manner, and by abridged multiplication of decimals; we obtain as follows:

$$7 + \tfrac{2}{12} = 7{,}1666\ldots\ldots : 6 + \tfrac{5}{12} = 6{,}41666$$

Performing the abridged multiplication thus:

$$\begin{array}{r} \dot{6},\dot{4}\dot{1}\dot{6}\dot{6}\dot{6} \\ \dot{7},\dot{1}\dot{6}\dot{6}\dot{6}\dot{6} \\ \hline 44,91666 \\ 64167 \\ 38500 \\ 3850 \\ 385 \\ 38 \\ 4 \\ \hline 45,98610 \end{array}$$

As both fractions are repeating, having a continued repetition of the 6, the products of these have been inserted, as long as they have any influence in the numbers preserved, by repeating the first product of 6, receding every time one place more to the right: and in the last numbers of the products always carrying from a product of a previous 6; this has been performed throughout, by augmenting the last figure one unit.

The second multiplication, treated in the same way, will, when executed, give the following

Example. $5 + \frac{7}{12} = 5,58333$, the series of 3 being again continued, multiplied in the last product gives

$$\begin{array}{r}
45,9\dot{8}\dot{6}\dot{1} \\
\dot{5},\dot{5}\dot{8}33 \\
\hline
229,9305 \\
22,9930 \\
3,6789 \\
1379 \\
138 \\
14 \\
1 \\
\hline
256,7556 \text{ \&c.}
\end{array}$$

To compare these three results together will be most easily done by reducing the two former ones to decimals. Thus we obtain by section 60,

$$256 + \frac{9}{12} + \frac{10}{1728} = 256,75578 \text{ \&c.}$$

by section 64, $$256 + \frac{9}{12} + \frac{5}{864} = 256,75578 \text{ \&c.}$$

by the decimals above, $= 256,7556$

The difference of nearly two units in the fourth decimal, or the ten thousandth parts, is owing to the neglect of the last decimals of the factors, which of course gives a smaller result, but which is in most cases sufficiently accurate; a farther extension of the decimals would of course cause a nearer approximation to the other results.

Examples. Execute under the three different forms, the following:

1*st.* 17 *f.* 9,17 *in.* × 12 *f.* 2,3 *in.* =

2nd. 41 *f.* 8,308 *in.* × 21 *f.* 11,481 *in.* =

3d. 3 *f.* 10,2 *in.* × 7 *f.* 4,32 *in.* × 9 *f.* 7,63 *in.* =

4th. 21 *f.* 8,897 *in.* × 42 *f.* 9,391 *in.* =

5th. 62 *f.* 7,95 *in.* × 0 *f.* 11,2 *in.* × 1 *f.* 6,3 *in.* =

6th. 36 *f.* 6 *in.* × 2 *f.* 3,5 *in.* × 1 *f.* 9,0 *in.* =

7th. 18 *f.* 3,5 *in.* × 3 *f.* 2 *in.* × 2 *f.* 2,5 *in.* =

8th. 21 *f.* 7,5 *in.* × 0 *f.* 9,5 *in.* × 0,9 *in.* =

9th. 24 *f.* 6 *in.* × 0 *f.* 10,5 *in.* × 0 *f.* 8,5 *in.* =

10th. 5 *f.* 6,5 *in* × 32 *f.* 6 *in.* × 24 *f.* 4 *in.* =

§ 66. DIVISION OF DENOMINATE FRACTIONS. The remark made in relation to the multiplication of this kind of fractions, upon the inconveniences of the operation, and its being applicable, generally speaking, to lineal dimensions only, applies still more forcibly to their division; in all other cases the quotients are not required in the same denomination of subdivisions; and in cases where the divisor and the dividend are of a different kind of quantity, they are in reality impossible in nature. But in the case of lineal dimensions, which produce by the first multiplication superficial magnitudes, and in a second solids, as both objects really exist in nature, it may sometimes be desirable to have a quotient given in the same denominate fractions, or subdivisions.

In all cases, therefore, where such a division occurs in the course of a calculation, the nature of the quantities concerned are left out of consideration, and the quotient inquired into is considered as a mere number. To do this, two ways present themselves; either to reduce every denomination of the numbers concerned to the lowest denomination of denominate fractions contained in them, and divide the whole numbers resulting; the quotient will give a result *in units of the whole* (not of the subdivision employed,) because *it expresses the value of the general fraction expressed by the division*, which is itself independent of the subdivision employed in the calculation.

Or the denominate fractions may be reduced into decimals of the whole quantity, as seen in the preceding section; and *the division of these will give exactly the same result, as the reduction to the lowest denomination;* because the quotient resulting gives also here the actual value of the fraction expressed by the division.

§ 67. *To perform a Division of Denominate Fractions.* As this may be desired, in lineal dimensions, it will be proper to give an appropriate general rule; as follows:

Find the whole number which, multiplied into the divisor, will give a product nearest below the dividend, and divide by it as in common division, only minding the transfer or carrying from one denomination to the other, according to the principle of the denominate fraction under calculation; then reduce the remainder to the next lower denomination, and multiply the product by the denominator of the denominate fraction; reduce also the whole divisor to the next lower denominate fraction, and divide the last obtained dividend by it; the result will give a number expressed in this next lower denomination; thus continue to the end of all the desired subdivisions.

The reason of this rule is evident, in its first step, from common division of whole numbers; in the second, and following steps, the multiplication of the remainder, after reduction by the denominator of the denominate fraction, is necessary to give by the division a result expressed in units of this lower division; in the same manner as for decimal fractions, a 0 was added to every remainder, to produce a quotient of the next lower rank of decimals, because this 0, produced a multiplication by 10.

Example. To divide

$$\begin{array}{l} \quad ft. \\ \dfrac{17\frac{9}{12}}{\;\;7.\ 10} = 2\ ft.\ 3\ in. + \frac{9}{47} \\ \hline 15.\quad 8 \\ \hline \;\;2.\quad 1 = 25\ in. \end{array}$$

multiplying this remainder by 12, and reducing also the divisor, gives the following second division:

$$\begin{array}{l} \dfrac{300\ in.}{94} = 3\ in. + \dfrac{12}{94} = 3\ in. + \dfrac{1}{8} \\ \overline{282} \\ \ \ 12 \end{array}$$

The first quotient found here, or the feet, is 2; then the remainder, 2 *ft*, 1 *in.*, reduced, gives 25 *in.*, which multiplied by 12, to give the next denominate fraction, by the division with the reduced divisor, or 94; this division gives 3 *in.*, and the fraction which is here left, but could be reduced again to twelfth parts, as the next subdivision, by the same operation as the inches were obtained, if desired. It must be observed here: that in the dividend the 9 were treated as twelfth parts of the square foot, which are not square inches; if they were such, they would present the denominator 12 × 12 = 144, as may easily be judged, by reflecting upon the multiplication shown above, or because 1 *ft.* =12 *in.* gives 1 *ft.* square = 12 *in.* × 12 *in.* = 144 square inches.

To execute the same Operation or Division by the two other Methods. The process of reduction to the lowest denomination, or to whole and decimal numbers, is evident from the principles of division taught for the two cases, in the division of whole numbers and of decimal fractions.

The above example would stand in them as follows:

1*st.* By reduction to the lowest denomination,

$$\frac{\overset{ft.}{17\frac{9}{12}}}{7.\ 10} = \frac{213}{94} = 2{,}2659\ ft. = 2\ ft.\ \overset{tw.}{3{,}1908}$$

this last by multiplying the decimals by 12, the denominator of the denominate fraction.

2*d.* By the whole numbers, and the value of the subdivisions in decimals,

$$\frac{17\frac{9}{12}}{7.\ 10} = \frac{17{,}75}{7{,}8333} = 2{,}2660 = 2\ ft.\ 3{,}192\ tw.$$

The first result will be somewhat too small, on account of the discontinued division, the second somewhat too large, on account of the discontinued series of 3 in the divisor, which, being smaller, leaves the quotient to become somewhat larger.

Examples. Execute in the three different methods the following divisions :

1st. $\dfrac{16\ \textit{f.}\ 7{,}32\ \textit{in.}}{2\ \textit{f.}\ 9{,}5\ \textit{in.}}$; *2nd.* $\dfrac{78\ \textit{f.}\ 42{,}32\ \textit{in.}}{3\ \textit{f.}\ 6{,}94\ \textit{in.}}$;

3d. $\dfrac{416\ \textit{f.}\ 27{,}9\ \textit{in.}}{44\ \textit{f.}\ 6{,}5\ \textit{in.}}$; *4th.* $\dfrac{302\ \textit{f.}\ 9\ \textit{in.}}{6\ \textit{f.}\ 7{,}5\ \textit{in.}}$;

5th. $\dfrac{212\ \textit{f.}\ 17\ \textit{in.}}{49\ \textit{f.}\ 6\ \textit{in.}}$; *6th.* $\dfrac{129\ \textit{f.}\ 9\ \textit{in.}}{22\ \textit{f.}\ 6\ \textit{in.}}$;

7th. $\dfrac{92\ \textit{f.}\ 9{,}5\ \textit{in.}}{50\ \textit{f.}\ 3\ \textit{in.}}$; *8th.* $\dfrac{105\ \textit{f.}\ 19\ \textit{in.}}{5\ \textit{f.}\ 10\ \textit{in.}}$;

9th. $\dfrac{99\ \textit{f.}\ 34\ \textit{in.}}{3\ \textit{f.}\ 2\ \textit{in.}}$; *10th.* $\dfrac{101\ \textit{f.}\ 84{,}5\ \textit{in.}}{2\ \textit{f.}\ 9{,}5\ \textit{in.}}$;

It is to be observed that the inches in the numerator are square inches, of which 144 make the square foot.

The whole of the examples executed in denominate fractions, and particularly the latter ones, show : that the calculation of denominate fractions properly belongs to the applied or practical part of arithmetic, which is intended to be treated in the next chapter ; it has however appeared proper to treat of their principles here, considering them as fractions of a particular nature, as these considerations tend to illustrate the general view of fractions ; to enter more minutely into details is however the province of the practical part, where more examples will appear, and where it will become evident to every attentive peruser of this work : that the proper understanding of the principles of arithmetic will suggest to him in every case the ideas which will lead him to the most judicious, accurate, and short way to execute calculations, implying such detail cases.

PART II.

PRACTICAL APPLICATIONS OF THE FOUR RULES OF ARITHMETIC.

CHAPTER I.

General Principles of the Application of the Four Rules of Arithmetic.

§ 68. In the previous chapters have been deduced, from the first principles of the combination of quantity and numbers, what are called the Four Rules of Arithmetic, and they have been applied to the different forms in which quantities are presented, namely: as *Whole numbers* of units, or as parts of the same; and these latter expressed either by their general relation to the unit, as in *Vulgar Fractions*, or by a continuation of the decimal system below the unit, as in *Decimal Fractions*, or as arbitrary subdivisions under the name of *Denominate Fractions.*

The preceding part may therefore be considered as the theory of the four rules of arithmetic; it will already present the solution of a great number of the questions arising in common life from daily intercourse or occupation. Though this application might be made by the teacher, it may not be improper, particularly for such persons as should wish to undertake the study of arithmetic by themselves, here to give a few leading ideas to guide them in the proper choice of the rule for any distinctly given case, together with some examples.

§ 69. Under the head of *Addition* will come: all such questions, where *quantities of the same kind* are to be counted together, as has been seen to be the origin of this first rule. It is of course impossible, to add *quantities of different kinds* together under any denomination than as mere *things*; and this remark, simple as it is, may escape in certain cases. We have seen, for example, in fractions, that we were compelled to make such changes in the denominators as produced the effect, of reducing the quantities which are of a different kind, on account of their being different parts of the unit, to quantities of the same kind, or denomination, before they could be added.

That all this applies equally to *Subtraction*, is evident from the principle, that it is only the opposite operation of Addition.

§ 70. So for *example*. A farmer, making the enumeration of his live stock, may add them under their different denominations, as different kinds, or in sum total.

Suppose, therefore, a farmer had his live-stock distributed in different lots of ground, as follows:

In the door-yard are 3 cows, each with a calf, 2 horses, and 4 pigs.

In the meadow he has 4 oxen, 6 cows, and 3 young horses.

In the field, a flock of 35 sheep, 5 cows and 4 calves.

He lets run in the woods, 9 pigs, 7 cows, 4 young oxen, and 2 horses.

We may ask here, first, the sum of all his live-stock, which will comprehend all what is above under one sum; thus:

2 horses
3 cows
3 calves
4 pigs
4 oxen
6 cows
3 horses
35 sheep
5 cows
4 calves
9 pigs
7 cows
4 oxen
2 horses
———
91 heads of live-stock.

2nd. We may ask how many of each kind, and then we shall have to separate the quantities above, in this manner:

Cows.	*Oxen.*	*Calves.*	*Horses.*	*Pigs.*	*Sheep.*
3	4	3	2	4	35
6	4	4	3	9	
5	—	—	2	—	
7	8	7	—	13	
—			7		
21					

Other examples of Simple Addition may be the following:

For a certain undertaking in a village, seven men agree to give, each as much money as he has in cash in pocket; John has $47; Peter $121; James $50; Richard $79; Francis $107; Frederick $192; and William $305; how much stock do they bring together? The addition gives:

$ 47
121
50
79
107
192
305
———
$ 901

How much is the whole banking stock in New-York, the stocks of the chartered banks being as follows:

Bank of New-York,	$ 950,000
Manhattan Bank,	2,050,000
Merchants' Bank,	1,490,000
Mechanics' Bank,	2,000,000
Union Bank,	1,000,000
Bank of America,	2,000,000
City Bank,	2,000,000
Phœnix Bank,	500,000
United States' Bank,	35,000,000
Franklin Bank,	500,000
North River Bank,	500,000
Tradesmen's Bank,	600,000
Chemical Bank,	500,000
Fulton Bank,	500,000

EXAMPLES.

1. The boys of this school have marbles in pocket, A has 12, B has 41, C has 17, D has 29, E has 34, F has 72, G has 15, H has 62, I has 90; how many have they all together?

2. From Savannah, in Georgia, to Charleston, is 118 miles, from thence to Raleigh, in North Carolina, 256 miles, thence to Richmond, 164 miles, thence to Washington, 123 miles, thence to Baltimore, 37 miles, thence to Philadelphia, 100 miles, thence to New-York, 100 miles, thence to Albany, 140 miles, thence to Whitehall, 70 miles, thence to Burlington, Vermont, 70 miles, thence to Montreal, 100 miles; how many miles is it from Savannah to Montreal? and if a man travels at the rate of 80 miles per day, how many days will he have to travel the whole?

3. The diameters of the principal planets are as follows: Mercury 16057000 feet, Venus 40125400 feet, the Earth 41836420 feet, Mars 21646200 feet, Vesta 212000 feet, Juno 12389460 feet, Ceres 2125480 feet, Pallas 27619700 feet, Jupiter 4643343000 feet, Saturn 41763300 feet, Uranus 181210000 feet; how much would they make in distance supposing them all placed together?

4. Rome being built 817 years after the Pyramids of Egypt, and 146 years before the Christian Era, America having been discovered 1594, the Declaration of Independence by the United States having taken place 182 years afterwards, and the year of it now counted being 52; how long is it since the building of the Egyptian Pyramids?

5. A captain has on board 170 bales, each paying freight $1,25; 305 packages, each paying 87½ cents; 230 tons of other goods, each ton paying $12,62½; and 6 passengers, each paying $78,50; how much does his whole freight and passage money amount to?

Ans. $3854,12½.

6. A grocer making an inventory, finds he has in cash $17,52; in various liquors the amount of $215,17; in soap, candles, and such articles, $92,54; in spices, $107,32; in salt fish and similar provisions, $49,62; and in various small articles besides the furniture of his store, in all $57,84; what is the whole amount of his stock?

EXAMPLES OF SUBTRACTION.

1. Francis has 35 head of cattle on his farm, and his neighbour James 84; how much has the one more than the other?

James's cattle	84	
Francis' cattle	35	
Difference	49	which James has more.

2. A man going into account with himself finds his whole property amounts to $18406, and that he has $10509 debts; how does he stand?

Property	$ 18406	
Debts	10509	
Difference	$ 7897	clear property left.

3. In a year there are 365 days; of these 52 are Sundays; how many working days are there in a year? *Ans.* 313 *days.*

4. A man in business bought, during the year, goods to the amount of $106409, and sold to the amount of $59879; taken at the same price or estimate, what amount of goods has he left? *Ans.* $46530.

5. What time elapsed between the publication of the Copernican system of the world, 1543, and Newton's first publication of his Philosophy, in 1686?

6. How many more inhabitants are there in the State of New-York, whose population is 1372812, than in the State of Kentucky, whose population is 564313?

7. As I have 76 sheep, and my neighbour has 135, how many has he more than I?

8. How long a time passed between the general peace of Europe made in 1666, and that made in 1815?

9. What is the present year of the Turks, as their era begins 622 years after the Christian era?

10. How long a time passed between the invention of gunpowder, in 1330, and the election of General Washington to the Presidency of the United States, in 1789?

11. The diameter of the planets being as stated in one of the preceding questions, how much is that sum less

than the diameter of the Sun, which has 4674790000 feet?

§ 71. The applications of Multiplication occur in every case where one of the quantities occurs repeatedly, this repetition being as often as indicated by another number, which forms the multiplier; such is the case, for instance, in all purchases, profits, interest, at a certain rate for the adopted unit of the things bought, sold, or lent, or in general, whenever the same thing or quantity is repeatedly taken.

EXAMPLES.

1. John buys 12 peaches at 3 cents a-piece; how much has he to pay? *Ans.* 36 *cents.*

2. 48 head of poultry bought at 5 cents per head; how many cents to pay? *Ans.* 240.

3. The year has 365 days, every day 24 hours; how many hours in a year? *Ans.* 8760 *hours.*

4. How many minutes are there in a month of 31 days, the day having 24 hours, and the hour 60 minutes?

Ans. 44640 *minutes.*

5. A merchant bought 56 bales of cotton goods; 15 of them held 21 pieces, 29 held 28 pieces, and the rest 25 pieces each; for each piece he pays $3; how much must he pay for the whole?

Ans. { $4281 to pay.
1427 pieces of cotton goods.

The numbers indicating the quantity of pieces in each bale are to be multiplied by the number of bales respectively; the sum of these results gives the whole number of pieces, which being multiplied by 3, the price of each piece, gives the final result.

6. A merchant bought 963 barrels of flour; on weighing them, he finds their average weight 202 lbs, and that the barrels average 7 lbs. weight each; how many pounds of flour has he? *Ans.* 187785 *lbs.*

Note. The subtraction needed in the above example from each barrel, is what in commerce is called *tare*; the remaining weight is what is called *neat weight.* Tare is usually determined by an approximate valuation, in each particular kind of package, according to certain habitual and even local rules. It is sufficient to know these, to execute any example of mercantile calculation relating to what is called *tare*, as they form a subtraction upon agreed principles.

7. The sum of $6500 is lent out at interest, for three years, at six per cent. simple interest, annually; what will be the whole amount of that interest in three years?

The interest per cent. being evidently a decimal fraction, in the place of the hundreds, or second decimal; the whole operation of any interest calculation for the year, consists in multiplying the capital given with the corresponding decimal fraction, and for more years, to multiply this product again by the number of years required; thus the above consists in the execution of the following multiplication:

$$6500 \times 0{,}06 \times 3 = 390 \times 3 = \$1170.$$

It is evident also from this, that all transactions of commission, brokerage, exchange, notes, drafts, stock, &c., which are grounded upon a certain per centage of premium, or discount, are exactly of the same nature, and determine a decimal fraction by which the amount is to be multiplied, as in the present example.

8. Seven boys have each twelve marbles; how many marbles have they all together?

9. If 5 boys buy each half a peck of apples, and each half peck holds on an average 16 apples, how many apples have they all together?

10. A company of soldiers of 105 men and the officers, having all muskets, each weighing 5 pounds, and 2 pounds of ammunition, how much weight have they to carry all together?

11. A ton, ship's weight, is 2200 pounds; how many pounds weight will be in a vessel carrying 450 tons?

12. Twenty bales of cloth, containing each 27 pieces, of 28 yards the piece, how many yards are there in the whole?

§ 72. *Division* applies in ordinary business to all cases, where any quantity of things is to be divided among an equal number of persons, or in an equal number of lots or parts; the quotient will give the share of each person, or the quantity of things in each lot or part. It will therefore also apply to find the price of a single piece of a thing, of which a large number has been purchased, for a certain price; as in the following

EXAMPLES.

1. A father, having six sons, leaves a property of $76590 to be shared equally among them; how much will each son get? *Ans.* $12765.

2. The provision of an army in bread is 90567 *lbs.*; it is intended to distribute the whole to the soldiers, to save separate transportation; there are 10063 soldiers; how many pounds will each have to carry? *Ans.* 9 *lbs.*

3. The expenses of paving a street, 500 feet in length, amount to $1000; the amount is to be distributed among the owners of the adjoining lots, each having a lot of 25 feet; how much will each lot or owner have to pay? *Ans.* $50.

4. A merchant bought 109 bales of calico, for the total amount of $12232; he finds that 40 bales contain each 30 pieces, 50 contain each 25 pieces, and the rest contain 32 pieces each; how high does each piece stand him? *Ans.* $4.

5. How many days are there in 24480 minutes, each day having 24 hours, each hour 60 minutes? *Ans.* 17*ds.*

6. How many days will it take a man to travel 945 miles, if he travel 35 miles per day? *Ans.* 27 *ds.*

7. I have 750 pieces of cloth, and can put no more than 15 pieces in a bale; how many bales shall I have to make?

8. A schoolmaster has 62 boys, and having a lot of 434 marbles, which he wishes to distribute equally among his boys as a reward; how many will each of them get?

9. If a man has an annual income of $3555, how much can he spend per day?

10. A man having two hundred and fifty miles to travel, and travelling 24 miles per day, how long will he be in performing the journey?

§ 73. It will be proper here to draw the attention to a general principle, which will always guide us in the use of multiplication, as applied to any purpose of life, or even of science; namely: it expresses always by the two factors a certain *cause* and a certain repetition of the same, which may be best represented by *time*, for this is the measure of repetition of effects in nature, as we have seen it to be, for instance, in calculations of interest, &c.; the result of these factors, giving the product of the numbers, represents equally well the *effect* produced by the cause, represented by the one factor acting a certain time, which is represented by the other factor.

So we may represent the multiplication and its results as *the product of cause into time*, being *equal to their effect*, (the great law which is to be exactly filled in every explanation or investigation of a subject of natural philosophy.) Instead of time, we may also call that factor power, and the other the object acted upon by this power; the ideas of cause and time however, always apply equally well; as for instance, a man having certain means to do a thing, and using them so much, or so many times, would be the same thing in respect to the effect; and so in all similar cases.

As the application of multiplication includes all the cases where a cause acting a certain time, or number of times, produced a certain effect; so division may, with equal propriety, be considered as *decomposing the effect, into its cause and time*; the one of these being given, besides the effect. Thus we evidently find: that if a certain work has been done, by a certain number of men, in a certain time, the work expressed in numbers representing the effect, the time of the work, or the number of men, being given, the number of men, or the time of their work, may be found, by dividing the effect inversely by the number of men, or the time they worked. In like manner: if the interest obtained upon a sum of money in a certain number of years be given, the yearly amount of it (that is the cause) will be given, by the quotient arising from the division of the whole amount by the number of years, and vice versa.

CHAPTER II.

Application of the Four Rules of Arithmetic to all kinds of Questions, involving Fractions of either kind.

§ 74. In most of the circumstances, where calculation is to be applied in common life, the given quantities either contain certain fractions, or denominate subdivisions of the unit, which have been called Denominate Fractions, or they often lead to such by division, as has been seen in its place. The calculator must determine by the aid of proper reflection upon any given case, and by his knowledge of the principles or theory of arithmetic, in what manner it will be most easy, and, according to the aim of the calculation, most accurate, to obtain the result. Practice gives great facilities for this determination; in the instructions for performing it, only general considerations, or principles can be presented, and examples that may serve as an introduction to it; this is the aim of the present chapter.

It may, for instance, be readily inferred from a comparison of the operations in Vulgar and in Decimal Fractions, that in complicated *additions* of numbers, involving vulgar fractions whose denominators are not simple, or commensurable, (that is, the one a multiple of the other,) the reduction of the fractional part into decimal fractions, before they are added, is peculiarly advantageous.

In *subtraction* the same is the case, in a less degree, however, on account of the circumstance of there being only two fractions that can possibly be engaged in one operation.

Denominate fractions present no difficulty, in either *addition* or *subtraction*, more than common numbers, except the attention that is necessary in the carrying, or borrowing, from one denomination to the other, but are from that circumstance, and their irregular progressions, far less convenient than decimal fractions. From this circumstance the decimal system derives great advantage, and has for that reason been introduced at least in the coins of the United States.

The reduction of *whole numbers* into fractions will never be needed in *addition*; when it may be required in subtraction, the application of the principle of borrowing one single unit, and reducing the same into the required fraction, as in the addition of whole numbers and decimals, will be the most advantageous and shortest method, wherever the reduction of the vulgar fraction into a decimal fraction does not present greater advantages.

The preceding remarks, in addition to what has been said upon the application of the two first rules of arithmetic, may suffice in this place; particularly as in multiplication and division they again naturally occur, and receive their explanation and application, in a manner still more instructive, than when treated alone.

A few examples placed here for exercise may therefore suffice; they will be expressed by the signs of the operation when given simply, in order at the same time to afford an opportunity of exercise in their use.

1*st*. Execute $17 + \frac{4}{7} + \frac{13}{4} + 3 + \frac{7}{8} + 9 + \frac{3}{8} =$

2*nd*. „ $\frac{3}{5} + \frac{5}{8} + 5 + \frac{5}{12} + 3 + 1\frac{1}{3} + 6 + \frac{7}{8} =$

3*d*. „ $\frac{7}{10} - \frac{4}{5} + 3 + \frac{7}{8} + 7 - \frac{3}{5} + 6 - \frac{7}{12} =$

4*th*. „ $5 + \frac{3}{4} + 2 - \frac{7}{15} - \frac{8}{13} + 7 - \frac{4}{19} + 3 +$
$\frac{5}{21} =$

5*th*. „ 4,65080906 + 0,0070606 + 114,604091
+ 0,985 + 406,307506 + 3000,040907 =

6*th*. „ 6,04097082 — 5,908986072 =

7*th*. „ 3,4091 — 3,064723 + 5,0809701
— 2,908062 + 101,01980 — 67,520998
+ 3,05 — 0,0672 =

8*th*. „ 13 *ft.* 7,5 *in.* + 4 *ft.* 6,3 *in.* + 16 *ft.* 0,5 *in.*
+ 0 *ft.* 7,13 *in.* =

9*th*. „ 6 *ft.* 9,2 *in.* + 0 *ft.* 11,6 *in.* — 6 *ft.* 4,25 *in.* =

10*th*. „ 3 *lb.* 7 *oz.* 2 *dwt.* 7 *gr.* + 5 *oz.* 7 *dwt.* 19 *gr.* =

11th. „ 107 *lb.* 4 *oz.* 0 *dwt.* 6 *gr.* + 5 *oz.* 6 *dwt.* *gr.* + 5 *dwt.* 19 *gr.* + 6 *lb.* 0 *oz.* 7 *gr.* — 106 *lb.* 10 *oz.* 15 *dwt.* 20 *gr.* =

Question 1. A farmer threshed grain 7 days:
the 1*st* day 12$\frac{1}{2}$ bushels
„ 2*nd* „ 18$\frac{1}{3}$ „
„ 3*d* „ 24$\frac{1}{5}$ „
„ 4*th* „ 30$\frac{2}{3}$ „
„ 5*th* „ 32$\frac{3}{4}$ „
„ 6*th* „ 44$\frac{2}{5}$ „
„ 7*th* „ 15$\frac{1}{2}$ „

He paid his help in grain; to one man he gave 3$\frac{1}{2}$ bushels, to another 2$\frac{1}{3}$; and he returned to his neighbour what he had last borrowed of him to go to mill, which was 7$\frac{3}{5}$ bushels; how much grain has he left? *Ans.* 165$\frac{4}{15}$ *bush.*

2. A man goes out to collect payment of bills; he pays also his own debts in going his round; thus he gets from James $77, 65; from William $105, 37$\frac{1}{2}$; then he pays his grocer $98, 12$\frac{1}{2}$; going on he gets from P. Jones $307, 62$\frac{1}{2}$; and from J. Johnson he gets $692, 875 now he thinks himself able to make a payment on his house of (with interest) $856, 625, and pays his tailor yet $28, 375; how much has he left when he gets home? *Ans.* $200, 4.

§ 75. We have seen that the multiplication of whole numbers alone, and of fractions alone, presents no difficulty, while the mixture of both, as we have given an example, by reducing the denominate fractions to fractional parts, with small denominators, equivalent to them, has shown an operation, which we would gladly have exchanged for one on the same principle as the decimal system. Still it will always depend upon the judgment of the calculator, to which mode he shall give the preference, if his data are partly given in fractions; because these are often, even more generally, in no complicated proportions, as for instance, $\frac{1}{2}$; $\frac{1}{4}$; $\frac{1}{8}$; $\frac{1}{3}$; $\frac{2}{3}$; $\frac{1}{5}$; and the like; and particularly when only one factor has a fraction, the operation may be easily enough performed, to permit the like reductions to be avoided, which may, for instance, in $\frac{1}{8}$; $\frac{1}{8}$; $\frac{3}{8}$;

and the like, lead to interminate decimals. In this, therefore, the judgment of the calculator must decide, and it is very improper to bind one's self to any single peculiar mode; reflection will lead to a calculation easy and accurate, while a mere mechanical process will, when a mistake occurs, cause embarassment. What we would here advise is, good order in all calculations; that any example, however complicated, be written distinctly and regularly, in the order in which it proceeds, accompanied by the signs of the operations that are appropriate, whenever the operation itself might not declare it distinctly. All this is nothing else but the necessary and well known principle, that every thing must be done with reflection and order, if it is to succeed.

The mode of proceeding will most likely be best elucidated by a few examples of different kinds, accompanied by appropriate reasoning.

1*st Example.* Seventeen packages of goods, each weighing $72\frac{1}{2}$ pounds; what is the total weight?

Ans. $1232\frac{1}{2}$ *lbs.*

2*nd.* If one hundred weight of wool is bought at $40\frac{1}{3}$ dollars, what will $17\frac{3}{4}$ hundreds cost?

Write this example thus:

$$\begin{array}{r} 17.\,\frac{3}{4} \\ 40.\,\frac{1}{3} \\ \hline 680.\phantom{\frac{11}{12}} \\ 30.\phantom{\frac{11}{12}} \\ 5.\,\frac{11}{12} \\ \hline 715.\,\frac{11}{12} \end{array}$$

The first product is obtained by the multiplication of 17 × 40; then $\frac{3}{4}$ × 40 gives the second line; then multiplying the $17\frac{3}{4}$ by $\frac{1}{3}$, or what is the same thing, taking the third part of it, the 17 gives 5, and the remaining whole quantity 2; which reduced to fourth parts, gives $\frac{8}{4}$ to be added to the fraction $\frac{3}{4}$, giving $\frac{11}{4}$; which are to be divided by 8; and give the fractional part $\frac{11}{12}$; the sum of all the three products, is the product of the whole numbers, and fractions, into each other, as required

3d. To give an example requiring more fractional operation, let the following multiplication be given:

following the operation as before, (the operation shall here be denoted by the signs.)

$$\begin{array}{r} 34.\frac{7}{8} \\ 19.\frac{3}{5} \\ \hline \end{array}$$

$$\begin{array}{r} 9 \times 34 = 306 \\ 10 \times 34 = 340 \end{array}$$

$$\left.\begin{array}{l}\text{Decomposing} \\ \text{the fraction } \frac{7}{8} \\ \text{into } \frac{1}{2}+\frac{1}{4}+\frac{1}{8}\end{array}\right\} \qquad \frac{7}{8} \times 19 = \left\{\begin{array}{lr} 19 \times \frac{1}{2} = & 9.\frac{1}{2} \\ 19 \times \frac{1}{4} = & 4.\frac{3}{4} \\ 19 \times \frac{1}{8} = & 2.\frac{3}{8} \end{array}\right.$$

$$\left.\begin{array}{l}\text{and the fraction} \\ \frac{3}{5} \text{ into } \frac{1}{5}+\frac{1}{5}+\frac{1}{5}\end{array}\right\} \frac{3}{5} \times \left(34+\frac{7}{8}\right) = \left\{\begin{array}{lr} 34\frac{7}{8} \times \frac{1}{5} = & 6.\frac{39}{40} \\ \text{the same} = & 6.\frac{39}{40} \\ \text{the same} = & 6.\frac{39}{40} \\ \hline \end{array}\right.$$

$$34\tfrac{7}{8} \times 19\tfrac{3}{5} = 683.\tfrac{11}{20}$$

In the same way any other example of mixed numbers would be carried on; this, which was chosen to show how to decompose the fractions for the operation, would evidently not be executed in this manner by a reflecting calculator; for the reduction to decimals is easy, and the decimal fractions terminate; thus:

$$\begin{array}{rr} 34\frac{7}{8} = & 34,875 \\ 19\frac{3}{5} = & 19,6 \\ \hline & 313,875 \\ & 348,75 \\ & 20,9250 \\ \hline & 683,5500 \end{array}$$

is multiplied with the greatest ease, and reducing backwards again; the fraction $\frac{55}{100} = \frac{11}{20}$,

$$\begin{array}{r|l} \multicolumn{2}{c}{5} \\ \multicolumn{2}{c}{\smile} \\ \frac{55}{100} & \frac{11}{20} \end{array}$$

4th. A lumber merchant has 12 pieces of timber of 40½ feet long, and 15 inches square; how many cubic feet has he; and what sum total will he get for it at $12\frac{1}{2}$ cents the cubic foot?

Here we may evidently proceed either by Denominate Fractions, and Practice, or Cross Multiplication, or by Decimals; for the latter the example presents the greatest facility, on account of the fractions being easily reducible; it shall be made here in the three ways, parallel to each other for the contents of each piece, the rest being in simple quantities, and the money of decimal division is in all cases best suited for decimal multiplication.

By Cross Multiplication.	*By Decimals.*	*By Practice.* ft. in.
46. 6	1,25	1. 3
1. 3	1,25	1. 3
11. 7. 6	1,25	1. 3
46. 6	250	$3\frac{3}{4}$
	625	
58. 1. 6		1. $6\frac{3}{4}$
1. 3	1,5625	46. 6
	46,5	
14. 6. 4. 6		46. 6 *in.* *ft.*
58. 1. 6	9,3750	23. 3 $= 6 = \frac{1}{2}$
	62,500	1. $11\frac{1}{4} = \frac{1}{2} = \frac{1}{24}$
72. 7. 10. 6	78125	0. $11\frac{3}{8} = \frac{1}{4} = \frac{1}{48} = \frac{1}{2} \times \frac{1}{24}$
	72,65625	72. $7\frac{5}{8}$

The cubic contents of each piece being, in decimal fractions, 72, 65625 feet, the 12 pieces give 871, 875 cubic feet, and these at $12\frac{1}{2}$ cents or 0, 125 dollars, bring the amount by multiplying 871, 875 × 0, 125, or, what is the same thing, 871, 874 × $\frac{1}{8}$ = 108, 984375 dollars; for which would in actual practice be given,

Ans. \$108, $98\frac{1}{2}$

§ 76. *The Division* of quantities, expressed in whole numbers and fractional parts, either vulgar or denominate, when the dividend only has fractions, can be made as in common numbers; after having divided the whole number, the remainder is reduced into a fraction of the same denominator as the fraction given, and the fractional part being added, the division is continued, and gives fractions of the same denominator; so may be continued as far as desired for the intended aim. For decimal fractions, the directions given (section 58) may suffice.

When both dividend and divisor are numbers mixed with fractions, it will be found the most satisfactory, therefore most generally the easiest, to reduce the whole numbers of both to the denominator of the fraction annexed to each; and writing the resulting fractions, execute the division as shown in its place.

When the divisor and dividend have both denominate fractions, we have seen (section 67) that the divisor in that shape is unwieldy and disadvantageous, and therefore we have there shown two methods applicable with nearly equal advantages, to which we therefore refer.

In the practical application, therefore, either the one or the other of these modes will be chosen, according as it presents the greatest advantages; for it is evidently useless to raise difficulties in a practical work of any kind, to have the pleasure or glory of solving them. The object here will therefore only be: to present such questions as are soluble by division, either alone, or combined with the preceding rules; for it must have been observed, that it is a general principle in arithmetic: that all the preceding rules are applied in any subsequent one; so in practical questions the same liberty must be allowed, and it is proper always to show, in any stage of a science, what can be done with what has been taught, up to the step which is making.

1st Example. To execute the division $\frac{13\frac{4}{5}}{6\frac{7}{8}}$ by reducing the whole numbers to improper fractions, and reducing the result, thus:

$$\frac{\frac{13\times5+4}{5}}{\frac{6\times8+7}{8}}=\frac{\frac{69}{5}}{\frac{55}{8}}=\frac{69\times8}{55\times5}=\frac{552}{275}=2+\frac{2}{275}$$

This operation, step for step, is sufficiently clear from the principles of fractions; the reduction of the whole numbers into fractions, with the addition of the numerators, gives a fraction to be divided by another, and multiplying both the numerator and denominator of this compound fraction by both denominators successively, the two whole numbers result, which are be divided for the final result. The application of this principle has no other difficulty in more complicated numbers, than the length of the multiplications; this example may therefore suffice.

In decimal fractions this example would stand as follows:

$$\frac{13,8}{6,875} = 2,00727272 \text{ \&c.}$$

2nd. To execute $\dfrac{354\frac{7}{9}}{84\frac{4}{11}}$ by both methods.

3d. To execute $\dfrac{752\frac{13}{26}}{62\frac{14}{15}}$ by both methods.

4th. A farmer has mowed $78\frac{1}{2}$ acres of meadow land, which yielded on an average $2\frac{2}{3}$ tons per acre, and besides having wintered with the hay 63 head of cattle, he has sold $62\frac{1}{3}$ tons; at what average per head has his cattle consumed the hay? *Ans.* $2\frac{1}{3}$ *tons.*

And how much hay did he make in the whole?

Ans. $209\frac{1}{2}$ *tons.*

5th. Six men undertake to make the hay on a piece of land for 65 dollars, besides their board, which they do in 11 days, and make 39 tons of hay; at what rate did they make wages, how much per ton did the hay cost to make, reckoning the board at 25 cents per day for each man, and how much hay did each man make on an average per day?

They made $\frac{13}{22}$ of a ton, or half a ton and $\frac{1}{11}$ per day.
" " $98\frac{1}{2}$ cents per day, nearly, besides board.
The hay cost \$2, 09 nearly, per ton to make.

6th. A man undertakes a job for \$195; he hires for help 5 men, at the rate of $62\frac{1}{2}$ cents per day and their board, which he calculates to cost him for each 25 cents daily; he works, with these men, 30 days; how much wages did he make per day, paying his board at the same rate as the men he hired? *Ans.* $1, 875 *per day.*

And how much did his men cost him? *Ans.* $131, 25.

7th. A sum of $1311 is to be paid in six equal instalments, with interest at 7 per cent. each time upon the sum unpaid, the first instalment being on the delivery of the goods, the others yearly. What will each payment amount to?

The *1st* will be $218, 5
„ *2nd* „ 294, 975
„ *3d* „ 279, 68
„ *4th* „ 264, 385
„ *5th* „ 249, 09
„ *6th* „ 233, 795

Dividing the whole sum by 6 gives the equal yearly payments of the capital, which subtracted every year from the last capital, gives the capital, the interest of which at 7 per cent. is to be added each year to the equal payments, the first or present payment having no interest upon it.

8*th.* There are in a page of this book on an average 2000 pieces of type; the space filled by them is 3, 1 inches broad, and 5, 5 inches long what square space in a mean does each piece occupy? *Ans.* 0, 008025.

EXAMPLES FOR THIS CHAPTER.

1. What do 5 pieces of cloth of 28½ yards each, come to, at $3, 37½ per yard?

2. One pound sterling is equal to $4, 444; (with continued decimals of 4;) how much is £975½, expressed in dollars? *Ans.* $4335, 55122.

3. A captain of a vessel has on board 706 packages, each measuring ⅛ of a ton; 89 others, each measuring ½ a ton; and 405 others, each measuring ⅓ of a ton; how many tons of lading has he? *Ans.* 267,3 *tons.*

4. A captain has on board 170 bales, each paying freight $1, 25; 805 packages, each paying 87½ cents; 230 tons of other goods, each ton paying $12, 62½; and 6 passengers, each paying $78, 50; how much does his whole freight and passage money amount to? *Ans.* $3854, 12½.

5. A raft contains 305 pieces of timber; of these 120 are oak, 36 feet long and 16 inches square; 50 pieces of oak, 45 feet 6 inches long, and 18 inches by 14 inches on the sides; 166 pieces of pine masts, reckoned at 2 feet 6 inches square and 60 feet long. The rest pine timber, 17 inches square by 50 feet in length. The oak timber sells at 45 cents per cubic foot; the masts at 80 cents the cubic foot, and the pine timber at 15 cents the cubic foot. How much money will the whole raft come to in the sale?

6. For plastering a wall the mason has to receive 21 cents per square yard (or the square of 3 feet each way, and containing therefore 9 square feet;) the wall which he has plastered is 13½ feet high, and 22 feet long; how much has he to receive for it? *Ans.* $6, 93.

And how many yards does the wall contain?

Ans. 33 *yards.*

7. How many square feet front of brick wall can be built with 36000 bricks, the thickness of the wall being the length of two bricks, and the end of the bricks being 4 inches by 2? *Ans.* 1000 *feet square.*

8. A merchant makes 16½ per cent. upon merchandize that costs him $7650; how much will his profit amount to? = 7650 ×,0, 165 = *Ans.* $1262, 25, (according to the principles of decimal fractions.)

9. The tare allowed upon a certain merchandize is 2½ per cent.; how much will it amount to upon 7355 weight?

(Expressed as above) = 183, 875.

10. A grocer had according to his last inventory 317 lb. 10 oz. of sugar; 561 lb. 4 oz. of coffee; 451 lb. 6 oz. tea; 15 lb. 3 oz. pepper; 3 oz. 6 dwt. mace; 152 lb. rice; 17 gallons rum. He has sold since, 283 lb. 6 oz. sugar; 341 lb. 7 oz. coffee; 349 lb. 5 oz. tea; 11 lb. 8 oz. pepper; 2 oz. 6 dwt. mace; 5 gallons and 3 gills of rum; 121 lb. 7 oz. rice; how much has he left of each kind?

11. A man has to travel 75 miles; he walks the first day 20 miles 3 furlongs; the second 18 miles 5 fur. 20 yds.; the third 23 miles 7 fur. 50 yds.; how much of his journey remains every evening to be performed?

12. William the Conqueror acquired the throne of England the 26th December, 1066, and died 8th September,

1087. His son, William the Second, who immediately succeeded, died the 2d August, 1100. Henry the First succeeded, and died the 10th December, 1135. How long did each of them reign?

13. Three men starting at the same time from one place, arrived at another determined place, the first after 10 h. 16 m.; the second after 12 h. 42 m.; the third after 15 h. 3 m. How much did each of them arrive after the other?

14. Bought 27 lb. 5 oz. 16 dwt. of drugs at the rate of $9, 75 the pound; how much will be the amount?

15. Bought 3 bales of cotton, the first weighing 1016 lb., the second 998 lb., the third 1093 lb., at $17\frac{1}{2}$ cents the pound; what is the amount to pay?

16. A room is 22 feet 5 inches long and 18 feet 9 inches broad; how many yards of yard wide carpet will it need?

17. A wall is 8 feet 7 inches high, and 65 feet 9 inches in circumference; how many feet of plastering will be in it?

18. Required the solid contents of a wall 74 feet 6 inches long, 2 feet 9 inches thick, and 24 feet 4 inches high?

19. Required the solid contents of a box 5 feet 2,5 inches long, 3 feet 5 inches broad, and 2 feet 5, 8 inches deep?

20. How many cubic feet of earth will fill a dock 205 feet long, 75 feet broad, and 8 feet 7 inches deep?

21. If 87 lb. 6 oz. of coffee cost $18, 38 what is the price of 1 pound?

22. What is the price per pound of spices, when 34 lb. 7 oz. cost $25,82?

23. What is the length of a piece of timber 15 inches square, the cubic contents of which is 69 feet 6 inches?

24. What must be the depth of a square vessel, 1 foot 3 inches one way, and 2 feet 2,5 inches the other way, that shall hold 4 feet 2,5 inches cubic measure?

25. What must be one side of an area containing 2015 square feet, when the other side is 50 feet 7 inches?

26. If a horse runs 8 times round a circus in 1 h. 45 m. 20 s., how much time will it need for each turn?

27. A lumber merchant bought 6527 cubic feet of tim-

ber, in 321 pieces: how much did each piece average in cubic feet?

28. A brick wall, 2 bricks length in thickness, is 69 feet long and 26 feet high; how many bricks does it contain, each brick being 8 inches long, 4 inches broad, and 2 inches thick, when laid?

29. A purchase of goods that cost $765,25, was sold for $973,52; what was the profit?

30. A man has $8264,91 debts, and his property amounts to $7431,80; how does he stand?

31. Three men buy land, the one 5,212 acres, at $2,25 per acre, the other bought 281 acres for $600, and the third bought as much land as they both, for $892; what had the first to pay, how much land did the second buy, how much land had the third, and at what price did it stand him?

32. A brick, when laid in the wall, has 7,8 inches length, 3,9 breadth, and 1,8 inches thickness; how many bricks will it take to build a wall two lengths of bricks thick, 25 feet long, and 36 feet high?

33. At 6 per cent. interest, what must be the capital that will produce an income of $500?

34. A man having $600 a year, how much may he spend a day to save $200 in the year?

35. What is the interest at 7 per cent. of $12,450?

36. Upon 20 hogsheads of sugar, of 850 lbs. each, what is the tare, at 3 lb. for every hundred weight?

37. Three persons purchase together $500 of stock, at 5 per cent. premium, which brings in 8 per cent. interest; how much must each pay, and how much yearly interest will each have for his share?

38. A house is to be plastered, at 21 cents per square yard. Now there has been plastered an entry 35 feet long, and 11 feet 6 inches high on both sides, the 2 ends being given in, as compensation for the vacancies on the sides. Two rooms of the same height, in each of which, two sides of 20 feet long are reckoned full, and one end of 18 feet also reckoned full, to compensate for the vacancies, the fourth side is given in. Two upper rooms of 20 feet long, 14 feet broad, and 9 feet high, are reckoned in the same manner as those below, and one room has 14 feet by

16 feet 6 inches, which is considered as plastered all round. How much will the expense of the whole plastering be?

39. Suppose the above entry and rooms were to be wainscoted with simple boards, at the rate of $1,25 for every hundred square feet, what would be the expense?

40. A quantity of goods is bought for $3,521, and sold at 15 per cent. loss, for what was it sold?

41. A dock to be filled in, has 250 feet length, 95 feet breadth, and the perpendicular depth being 8 feet on an average, how many cart loads of earth are needed to fill it, at the rate of 7 cubic feet per cart load; and how much will it cost at 6 cents per load?

42. How much will the glazing of a house cost, that has 28 windows, each of 24 panes of glass, at the rate of $13\frac{1}{2}$ cents for each pane?

43. How many bricks are there in a wall two lengths of a brick thick, 20 feet long, and 38 feet high, the bricks being of the dimensions stated in the thirty-second question?

44. An old tower 40 feet square on the outside, has at first, a wall 10 feet thick for 20 feet of elevation, then for 36 feet the wall is 8 feet thick, then for 16 feet it is 5 feet thick, the outer sides being perpendicular; how many cubic yards of stone are there in these walls, (neglecting doors and window openings) how much will the stones cost, at 22 cents the cubic yard, and how much will the building of the wall cost, at the rate of 29 cents for every cubic fathom? What will be the weight of stones in it, the cubic foot being reckoned at 178 lbs.?

45. A carpenter has $6\frac{1}{2}$ cents per cubic foot for hewing timber; now he hewed 25 pieces of 15 inches square, (on each side) and 36 feet long; 16 pieces of 1 foot each way, and 42 feet long; 28 pieces 18 inches by 20, and 26 feet long; 12 pieces of 10 inches each side, and 32 feet long; and 15 pieces of 8 inches by 12 each side, and 18 feet long. How much money has he earned?

46. Two rooms are to be painted all round, the height of which is 12 feet 4 inches, the length of one, 32 feet, and its breadth 24 feet; the other, 18 feet 6 inches long, and 16 feet 5 inches broad, how much will be the cost, at 7 cents per square yard?

47. What will be the expense of paving a street 563 feet long, and 30 feet wide, at the rate of 65 cents per square yard?

48. What will be the weight of lead that is upon a roof 25 feet long, and 28 feet 6 inches slant on each side, at the rate of $8\frac{1}{2}$ lbs. the square foot?

49. What will be the amount of slating a roof of 38 feet 6 inches long, 31 feet 4 inches slant on each side, at the rate of $4,25 per square, of 10 feet side?

50. How many days will three carpenters take to shingle a roof 88 feet long, 28 feet slant on one side, and 32 on the other, at the rate of two and a half square, of 10 feet side, per day for each man, and how much will it cost at $1,20 per square?

51. What will be the amount of 4572 square feet of boards, at the rate of $10,50 per thousand feet?

52. A vessel imports goods to the amount of $9650, which pay duties at 21 per cent, on their value; of $12,600, paying 30 per cent.; and of $21580 pay 15 per cent. duty; besides 30 casks of wine, averaging 58 gallons, each of which pays 20 cents per gallon; what will the duties on the whole cargo amount to?

53. How many miles did that vessel travel in a year, which made three times the voyage to Europe and back again, every time averaging 26 days, sailing in a mean, at the rate of $6\frac{1}{2}$ miles an hour?

54. If a voyage to Batavia takes 90 days, the vessel sailing on an average $5\frac{3}{4}$ miles an hour, how many miles does the vessel sail in the whole voyage?

55. If a baker works out 9 barrels of flour every working day in the year, at 196 lbs. each barrel, how many pounds of flour does he use, and if he make one third more weight of bread out of it, how many pounds of bread does he make, and if he sells the bread at 4 cents the pound, how much does he make in a year, when the flour costs $5 per barrel?

56. If 18 dozen bottles of wine cost $62, what is the price of each bottle?

57. The nearest approximation between the earth and Venus, is in a mean 32,560,000 miles, the velocity of a cannon ball being about 2000 feet in a second, how long

would the cannon ball have to run, to go from planet to planet, if they remained stationary in such a position?

58. A year's rent of a house being $96, the occupant has laid out in repairs $24,56, and paid the taxes amounting to $7,45, what has he yet to pay?

59. A man having $660 a year, economises $150 annually; his income being raised to $1500 a year, how much can he spend daily to economise double as much as before?

60. If a man earns 65 cents per working day, at what price can he board, so as to save $89 for his clothing and other expenses per year?

61. A bill of Exchange on London for 372*l.* 12*s.* sterling, is bought at 8 per cent. premium, what is to be paid for it in dollars, at $4,44 to the pound sterling?

62. What will the commission at $2\frac{1}{2}$ per cent. amount to, on goods of the amount of $7652?

PART III.

OF RATIOS AND PROPORTIONS.

CHAPTER I.

Elementary Considerations of Ratio.

§ 77. In the very outset we have shown, that quantity was all that is capable of increase or decrease, without regard to the nature or kind of things the number of which was increased or decreased. From the simple step of considering two things together, or adding them, and then successively more, and the same a repeated number of times; or diminishing a certain number of things, for as many as a given number indicates, first once, and then as many times as it may admit of, both by the use of a determined system of numeration; we have arrived step by step at the *principles of the combination of quantity*, and conversely again, to the *decomposition of a combination into its parts*.

This process has led us to the four rules of arithmetic, that have been explained successively, two of which have been shown to be the opposite of the two others; each leading alternately to the decomposition of the composition of the other, as addition and subtraction, multiplication and division; the latter two of which have been shown to be the result of the continued repetition of the first two. By these means all the operations upon quantity in usual life, which depend merely on combinations, have become calculable, as shown by the application made of this theory in the second part.

This retrospective view of the part of arithmetic hitherto treated of, appears proper to be taken here, in order to awaken appropriate reflections in reference to the whole of what has been done, and the means it has furnished for further progress. The scholar, attentive to what he has done hitherto, cannot but have acquired the faculty of reasoning upon quantity. The reflections which we shall have to make in future will be as simple as before, but the application of them will require that he have made himself acquainted with the tools, or means, which he has been shown in the first part and has now to use in the following parts of arithmetic, and that he have acquired some dexterity in their use; he will do well therefore to cast back upon the whole a cursory view, in order the better to comprehend the general ideas that have directed it.

§ 78. The consideration which will be the foundation of the part of arithmetic to be now treated of, is the *Relation* which the quantities may have to each other, whether they be combined in any way, or not.

The relation of quantities to each other, in whatever way it may be, is called their *Ratio*. As we have seen that the increase or decrease of quantities depends on their combination, so their relation to each other, that is, their *Ratio*, must also depend on their possible combination, as it is determined by it. The *Ratio* is, therefore, also considered in relation to these combinations; and, as we have had the two principal combinations, of addition or subtraction, and of multiplication or division, so we have also two *kinds of ratio*, corresponding to them; namely, by *addition or subtraction*, and this is called *Arithmetical ratio*; and by *multiplication or division*, which is called the *Geometrical ratio*. We evidently here again find the second a repetition of the first, as multiplication and division are the repetition of addition and subtraction; but we may omit going so far back into elementary considerations, and proceed forward with the general idea, to render it fruitful for practical use.*

These two kinds of ratio take their mark of notation from the marks applied to the combinations or rules of arithmetic, on which they depend; thus:

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* The propriety of these denominations is not worth discussing; they are mere names, to which the idea above explained is to be attached, which forms what is called their definition

The arithmetical ratio of 7 to 3, is expressed by 7 — 3

The geometrical ratio of 7 to 3, is expressed by

$$7 : 3 ; \text{ or } \frac{7}{3}$$

They might be equally well expressed by the signs of addition and multiplication, if we were in the habit of generalizing the considerations on quantity to that extent; and we shall see hereafter, that their theory leads to it; that is to say, that when the ratio of two quantities by subtraction, or division, to which the above signs are appropriated, are given, their ratio by addition, or multiplication, is also given; or the one is a consequence of the other. In the habitual mode of writing, therefore, an *arithmetical ratio* expresses *a Difference* between two quantities, and a *geometrical ratio* expresses *the Quotient* arising from the division of the two quantities; this latter is called the *Index*, when referred to the geometrical ratio.

§ 79. The simplest reflection leads to the idea: that two or more such ratios may be exactly equal to each other, as well as two quantities in general; such an *Equality of ratio* is called a *Proportion.*

This principle between two ratios is expressed very naturally by the sign of equality between them, as for *example*:

An arithmetical proportion will be expressed thus:

$$7 - 3 = 12 - 8$$

This says: the difference between 7 and 3 is equal to the difference between 12 and 8.

A geometrical proportion will be expressed thus:

$$12 : 3 = 16 : 4 ; \text{ or } \tfrac{12}{3} = \tfrac{16}{4}$$

And this says: the quotient of 12 divided by 3, is equal to the quotient of 16 by 4, as it is evidently in both ratios = 4; and this is therefore also the *Index* of the two equal ratios.

The *first term* of a *ratio* is called the *Antecedent*, the *second* the *Consequent*; the first and last terms of a *proportion* are called the *Extreme terms*, the second and third the *Mean terms*.

A nearer investigation of the properties of these ratios will justify the assertion made above, for we shall find: that the arithmetical proportion, expressed as a difference, gives also an equality of sums; and the equality of the quotients or *Indices*, in the geometrical proportion, an equality of products; and that in this property lies their extensive utility in all calculations.

§ 80. It may be easily seen that, while in the preceding part of arithmetic, grounded upon combination only, we were limited to things of the same kind. We obtain by this extension, or the consideration of the *Relation of two things* to each other *in respect to Quantity*, the means of forming conclusions by calculation from things of different nature mutually acting upon each other, or whose quantity depend on each other; by the condition, or simple consideration, of *the Equality of the ratio of two things of one kind, to two things of another kind*, which we observe in nature in all things; for we may see a herd of cattle, as much, or as many times, larger than another herd of cattle, as the money owned by one man is as much, or as many times, larger than the money owned by another man; a mountain as much or as many times higher than a house, as the amount of one bill of exchange is of as much, or as many times, a greater amount than another; always with reference to a determined unit, for each kind of things, which is understood or designated, by the denomination.

These considerations are daily made in common life, by every one, and they need only be transferred into the language of arithmetic, to direct us in the principles of calculation derived from them.

The first of these ratios and proportions, namely the arithmetical, are naturally more limited in their application to practical purposes, as they are the result of a more limited scale of combination. The applications of the second, namely, the geometrical, are much more extensive, depending on a higher scale of combination; the geometrical proportion is the principle of what is called in arithmetic the *Rule of Three.*

§ 81. I have thought proper to enter into these elementary deductions, though their aim is thereby kept back for a short time, because it is all-important in any study to conceive the fundamental ideas in their generalization, thereby the explanation is so much facilitated, as ultimately to lead to a shortening of the task, both of teaching and of studying. To render these fundamental ideas useful, we shall in the first place show the consequences which lie in them, from the principles of combination upon which they are grounded, and the condition of equality, which forms the particular nature of a proportion. We may already, from the simple enunciation in signs, as it appears above, conclude: that their application to practice consists in the evident property, that any three of the four quantities so conditioned, being given, the fourth is necessarily determined; the manner in which this is rendered of practical use, will appear from the investigation of the *Properties* resulting from the *principles of these proportions.*

CHAPTER II.

Arithmetical Proportion.

§ 82. In *Arithmetical Proportion* the principle evidently is: that the difference (or, as shown equally well, the sum) of two quantities be equal to the difference (or sum) of two others. Therefore if each ratio is increased or decreased by the same quantity, the *Principle* of equality continues to subsist as before, because the quantities employed, and the ratios themselves, are both equal; as it is evident that the arithmetical proportion expresses only an equality of two quantities in the form of the difference (or the sum) of the two others; thence we have, for instance, from the preceding arithmetical proportion,

$$7 - 3 = 12 - 8$$

by adding on both sides the number 8,

$$7 + 8 - 3 = 12 - 8 + 8$$

and by again adding 3 on each side,

$$7 + 8 - 3 + 3 = 12 - 8 + 8 + 3$$

And as we have seen that addition and subtraction are inverse operations, and therefore compensate each other, the + and — also compensate, when they are affixed to the same quantities; therefore the $+3-3$ on one side, and the $+8-8$ on the other, reduce both these numbers to nothing, and our arithmetical proportion is changed by it into

$$7+8=12+3$$

an expression exactly of the kind that it has been said (section 78) could also be used for expressing the arithmetical proportion; this result, expressed in words, gives the *fundamental property of arithmetical proportions*, that: *in any arithmetical proportion, the sum of the two extreme terms is equal to the sum of the two mean terms.*

If we had expressed the arithmetical proportion as a sum, as the above expression shows, we would have the result: that *the difference of the extremes is equal to the difference of the means*, by the simple principle of the two arithmetical operations of addition and subtraction being opposite to each other.

Or, we would obtain from

$$7+8=12+3$$

by subtracting 3 on each side,

$$7+8-3=12+3-3$$

and by subtracting in this 8 on each side,

$$7-3+8-8=12+3-3-8$$

where we obtain again by the compensations, as shown in the other case,

$$7-3=12-8$$

that is, the arithmetical proportion in the form in which it was first stated.

From the above result we are authorised to conclude: that any operation of arithmetic, performed equally on both equal ratios, leaves the principle of the equality of the ratio unchanged; that is, equality will exist between them notwithstanding; and by this principle are guided, and of course deduced with full authority, any changes in the parts that may become necessary for a given aim, in practical calculation; thus it is evidently allowable to make the following *changes in the above arithmetical proportion;*

As: original, $7 - 3 = 12 - 8$
by adding equals $7 - 3 + 6 = 12 - 8 + 6$
subtracting equals, $7 - 3 - 2 = 12 - 8 - 2$
multiplying by equals, $5 \times 7 - 3 \times 5 = 12 \times 5 - 8 \times 5$
dividing by equals, $\frac{7}{4} - \frac{3}{4} = \frac{12}{4} - \frac{8}{\text{[illegible]}}$

Upon the same principles the places of the terms may be interchanged, by transposing the two extremes, or the two means, or both, mutually; either of the proportions resulting will give the sum of the mean terms equal to that of the extremes, that is, preserve the fundamental principle of presenting an equality of differences, as well as in the original proportion; thus is obtained:

$$7 - 12 = 3 - 8$$
$$8 - 3 = 12 - 7$$
$$8 - 12 = 3 - 7$$

all giving $7 + 8 = 12 + 3$

Two or more such proportions may also be composed by the addition or subtraction of the terms, respectively term for term; thus the following two arithmetical proportions will give results as follows:

$$\left.\begin{array}{l} 7 - 3 = 12 - 8 \\ 16 - 4 = 19 - 7 \end{array}\right\} \text{being given,}$$

we have from them by addition and subtraction,

$$\left\{\begin{array}{l} (7 + 16) - (3 + 4) = (12 + 19) - (8 + 7) \\ (16 - 7) - (4 - 3) = (19 - 12) - (7 - 8) \end{array}\right.$$

which gives again the sum of the extremes equal to the sum of the means, that is the fundamental principle of this proportion.

§ 83. In these principles and combinations, or mutations, lie the means, by which numbers, thus related to each other, are made susceptible of calculation; their mutual dependance therefore shows: that, when any three of them are given, the fourth is necessarily determined, therefore calculable, according to the principles here explained. The arithmetical process resulting from them is evident, for if from one of the above sums of extremes or means, we subtract either of the terms of the other sum, we shall have the result equal to the other term of the latter sum, or what is called the fourth term. Thus, if we had in the above original arithmetical proportion the first three terms given, as

$$7 - 3 = 12 - 8$$

making the sum of the mean terms,

$$3 + 12 = 15$$

and subtracting from it the first (or the given) extreme, namely, 7, we have

$$15 - 7 = 8$$

and then the complete proportion

$$7 - 3 = 12 - 8$$

as above.

§ 84. When, in such an arithmetical proportion, the same number which has been *the consequent in the first ratio, is the antecedent of the second ratio*, the proportion is called *a Continued arithmetical proportion;* as in the following:

$$12 - 10 = 10 - 8$$

The middle term, which is repeated, is called *the metical Mean;* it is of course equal to half the sum of the two extremes; we have, for instance, here

$$12 + 8 = 10 + 10 = 2 \times 10$$

or $$\frac{12 + 8}{2} = \frac{20}{2} = 10 = \frac{2 \times 10}{2} = 10$$

that is, identical results.

Such a proportion may evidently be continued through a whole series of numbers, as follows:

$$12 - 10 = 10 - 8 = 8 - 6 = 6 - 4 = 4 - 2 = 2 - 0$$

Then the numbers 12; 10; 8; 6; 4; 2; 0; are said to be in *continued arithmetical proportion;* and the *Series*, thus resulting, is called an *arithmetical Progression*, or an *Arithmetical Series*. Their use is very frequent in higher calculations, and we shall treat of them hereafter; we will here only state, that the successive numbers may either *increase* or *decrease* according to the same principle; and that, from the nature of their application in practice, they are always written in the manner we have stated that arithmetical proportion might be written; namely, with the sign of addition between the terms; thus

$$12 + 10 + 8 + 6 + 4 + 2 + 0$$

would be a *Decreasing arithmetical progression, or series;* and

$$3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$$

an *Increasing arithmetical series,* or *progression;* both are subject to the same laws, and the same principles, for the mutual determination of their several parts from each other, as we shall see in its proper place.

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CHAPTER III.

Geometrical Proportion.

§ 85. The principles of *Geometric Ratio,* as we have seen above, take their rise in the combinations of the second kind, explained in Part I., that is, *from Multiplication* or *Division.* In it therefore the ratio is considered as the indication of *how many times* a quantity is greater or smaller than another; the quantity indicating this ratio in one single number is called the *Index of the Ratio;* it is exactly *the same as the Quotient in a fraction or in a division.*

The investigation of the consequences of this principle in a geometrical proportion gives the general law, which must guide all the operations founded upon geometrical proportion, and lead to the discovery of all its properties. For this purpose it is habitual to present the geometric proportion as an *Equality of fractions, or quotients,* which we have found it to be; thus we have

$$12 : 3 = 16 : 4$$

or $\frac{12}{3} = \frac{16}{4}$; evidently presenting the identity $4 = 4$ by the execution of the division, and indicating 4, as the *Index* or the *Quotient.*

Reducing the two fractions to a common denominator, we obtain, without any change in the value, (as proved in fractions)

$$\frac{12 \times 4}{3 \times 4} = \frac{16 \times 3}{3 \times 4}$$

On account of the whole fractions or quotients being equal, from the nature of geometric proportions, and at the same time also the denominators of the fractions obtained, which are identical, being the products of equal factors, it is a necessary consequence, that the numerators must also be equal.

Therefore $12 \times 4 = 16 \times 3$

which is evidently identical with $48 = 48$

Comparing this result with the geometrical proportion given, we obtain the proof of the *essential property of geometrical proportions*; that *the Product of the two extreme terms is equal to the Product of the two mean terms.* A property exactly analogous to that obtained for the arithmetical proportion, which in that case relates to the sums of the terms, and in this to the products.

This at the same time confirms the general principle stated above: that a geometric proportion might be equally well expressed by a product, as by a quotient, and by operations the converse of those made above, it would lead to the expression of an equality by division, that is, equal quotients or ratios. We would in that way of representing the proportion, by dividing both sides successively by 3 and 4, obtain from the expression

$$12 \times 4 = 16 \times 3$$

$$\frac{12 \times 4}{3 \times 4} = \frac{16 \times 3}{3 \times 4}$$

And because the 4 in the one fraction, and the 3 in the other fraction, compensate, by division in the numerator and denominator, we have from this:

$$\frac{12}{3} = \frac{16}{4}$$

or $12 : 3 = 16 : 4$

that is, the identical expression of the usual geometrical proportion.[*]

§ 86. The principle now deduced, and proved, gives all the consequences, which are so useful in the application of geometrical proportions to practical calculation; namely: that in a geometrical proportion, *all those mutations are admissible, which do not alter the principle, that the product of the two extreme terms is equal to the product of the two mean terms.*

Therefore we can make all the changes analogous to the arithmetical proportion; in relation to the example before us, we deduce from

$$12 : 3 = 16 : 4$$

1st. By transposing the

$$\left.\begin{matrix}\text{middle}\\ \text{extreme}\end{matrix}\right\}\text{terms}\quad\left\{\begin{matrix}12 : 16 = 3 : 4\\ 4 : 3 = 16 : 12\end{matrix}\right.$$

2nd. Changing antecedents into consequents

$$3 : 12 = 4 : 16$$

These are all evident, from the simple principle: that the products of two quantities are the same, whichever of the two be the multiplier, or multiplicand; that is, because 3 times 4 is the same as 4 times 3, as well known; or any two other numbers; the above all equally present:

$$3 \times 16 = 4 \times 12 = 48.$$

3d. Multiplying by the same number either

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[*] The mathematical expression of those two modes of presenting the geometrical proportion would be; by the *products*; that *the proportion is an equality of products*; and by the usual mode it is: *a proportion is an equality of quotients*; this cannot escape the notice of any one reflecting upon the principles stated at the very outset: that in all the arithmetical principles of operation, the system must hold good, or be true, both directly and conversely. The sign of equality between the two ratios forming a proportion, is therefore the only proper sign, and the four dots used by many authors, are against principles, because they do not convey the idea of the principle, a thing so essential to actual knowledge. So to write, for instance, 12 : 3 : : 16 : 4 is wrong, or at least, a pleonasmus of signs, leading into misapprehensions, a thing contrary to principles in exact science.

both antecedents, as $2 \times 12 : 3 = 2 \times 16 : 4$
or both consequents, as $12 : 2 \times 3 = 16 : 2 \times 4$
or all the terms, as $2 \times 12 : 2 \times 3 = 2 \times 16 : 2 \times 4$

The results must evidently preserve the principle of equality of products of extremes and means; because in every case the same multiplier is contained in each product; for, though the products present other numbers, the identity of their results reduces them to the same principle.

4th. Dividing in the same manner as before will give in the same order:

$$\tfrac{12}{2} : 3 = \tfrac{16}{2} : 4$$
$$12 : \tfrac{3}{2} = 16 : \tfrac{4}{2}$$
$$\tfrac{12}{2} : \tfrac{3}{2} = \tfrac{16}{2} : \tfrac{4}{2}$$

To which the same reasoning applies as to the multiplication; and it is proper to make the division in all cases, where the data of a proportion are compounded of numbers having common measures, in the corresponding terms; that is, either in the two terms of one or the other ratio, or in both antecedents, or in both consequents.

We can also compose and decompose the geometrical proportion by its antecedent and consequent terms, in such a manner as to obtain the proportion between their sum or difference and the antecedents or consequents, or between these sums and differences themselves, which furnishes an additional means of calculation for a number of practical cases.

5th. Thus we obtain from our example the following results of mutations; viz:

By adding the antecedent and consequent and comparing them to the antecedents:

$$12 + 3 : 12 = 16 + 4 : 16$$

By comparing the same to the consequents:

$$12 + 3 : 3 = 16 + 4 : 4$$

By comparing the differences of the antecedents and the consequents to the antecedents:

$$12 - 3 : 12 = 16 - 4 : 16$$

By comparing the same to the consequents:

$$12 - 3 : 3 = 16 - 4 : 4$$

By comparing these sums and differences themselves:

$$12 + 3 : 12 - 3 = 16 + 4 : 16 - 4$$

All these compound proportions have necessarily the property of giving equal products of the extreme and the mean terms, and all the mutations under *3d*, *4th*, and *5th*, again admit, of course, the exchange of the places of the extreme and the mean terms, which the original proportion admits.

Therefore, we may also, instead of adding or subtracting antecedents to or from consequents, and alternately to compare to either consequents or antecedents, or one to another, add to each other, or subtract from each other immediately, the two antecedents or the two consequents, and compare them exactly in the same manner as heretofore. As for example:

from $12 : 3 = 16 : 4$

deduce $12 + 16 : 3 + 4 = 16 : 4 = 12 : 3$

or $12 + 16 : 16 - 12 = 3 + 4 : 4 - 3$

Which will give by products of extremes and means:

$$\text{the first} \begin{cases} (3 + 4)16 = (12 + 16)4 \\ 112 = 112 \end{cases}$$

$$\text{the second} \begin{cases} (4 + 3)(16 - 12) = (4 - 3)(12 + 16) \\ 28 = 28 \end{cases}$$

and so in the other corresponding mutations.

Any one of these mutations is to be applied either to disengage one of the quantities contained in a given proportion, or whenever it can lead to an abridgement of the statement; and in proportions apparently compound they often lead to the final solution, without its being necessary to have recourse to both the multiplication and division of the terms themselves, only the one or the other of the operations remaining at last to be performed; that is, the one or the other term is reducible by it to unity; the future application will show this by examples in given cases.

As any of these operations can be repeated again, with every new proportion resulting, a proportion can be obtained, between any number of repeated sums and differences of antecedents and consequents, compared either one to the other, or to either of the antecedents and consequents of the original proportion; and as the consequents can be changed into antecedents also to the antecedents, or to the consequents.

§ 87. If we have two geometrical proportions, they may be multiplied together, or divided the one by the other, term by term, with equal correctness of conclusion; for it is the same as multiplying two equal fractions by two other equal fractions, the products of which will again be equal; therefore, according to the principles first deduced, the products of the extreme and mean terms will again be equal.

For example, let the two following proportions be thus composed; viz:

$18 : 6 = 12 : 4$ or the fractions $\frac{18}{6} = \frac{12}{4}$

and $15 : 3 = 25 : 5$ " " $\frac{15}{3} = \frac{25}{5}$

Multiplying the proportions term by term, or equal fractions by equal fractions, we obtain:

$$18 \times 15 : 6 \times 3 = 12 \times 25 : 4 \times 5; \text{ or } \frac{18 \times 15}{6 \times 3} = \frac{12 \times 25}{4 \times 5}.$$

where the product of extremes and means gives

$$18 \times 15 \times 4 \times 5 = 6 \times 3 \times 12 \times 25$$
$$5400 = 5400$$

and by reducing the fractions, by means of their common measures:

$$15 = 15$$

In like manner, by division, we would obtain from the foregoing

$$\frac{18}{15} : \frac{6}{3} = \frac{12}{25} : \frac{4}{5} \quad \text{or} \quad \frac{18 \times 3}{6 \times 15} = \frac{12 \times 5}{4 \times 25};$$

giving, by products of extremes and means,

$$\frac{6 \times 12}{3 \times 25} = \frac{4 \times 18}{5 \times 15} = \frac{24}{25} = \frac{24}{25}$$

or as fractions $\frac{3}{5} = \frac{3}{5}$

all equally leading to identical results.

Supposing, therefore, six terms in these two proportions given, in any manner, the two remaining terms may be determined from them. And in general: as many proportions as are given, so many unknown quantities may be determined by them.

This is the principle of what is called in arithmetic *the Compound Rule of Three.* It may be carried to any length, by further combination upon the same principles. When it is carried through a number of proportions, to determine only one unknown quantity, it is called *the chain rule.* The application of both, and their extensive utility, will be shown in their proper places.

The proportion may also be multiplied *into itself* term by term; and thereby may be obtained, from the proportions of lineal dimensions, the proportion of the superficial dimensions corresponding to them, or the squares. By the products of three such equal proportions term by term, will be obtained the proportion of the solids having the same lineal dimensions for their sides, or the cubes. Thus would, for instance, be obtained:

From the simple proportion $18 : 6 = 12 : 4$

the square, $18 \times 18 : 6 \times 6 = 12 \times 12 : 4 \times 4$

or $324 : 36 = 144 : 16$

the cubes,

$18 \times 18 \times 18 : 6 \times 6 \times 6 = 12 \times 12 \times 12 : 4 \times 4 \times 4$

or $5832 : 216 = 1728 : 64$

§ 88. It may readily be conceived: that in geometrical proportions a continuance may take place, as well as in the arithmetical; that condition may be again expressed by the equality of the two middle terms; as follows:

$16 : 8 = 8 : 4$; which gives $8 \times 8 = 16 \times 4$

as the products of extremes and means. The middle term is called *the Geometrical mean.*

To this every property applies that belongs to general proportion; it therefore admits all the changes heretofore shown. The product of the two mean terms, being compounded of two equal factors, presents what is called a square number; comparing it by this to the rectangular surface which would have all its sides equal, and showing by the equality of the products thus obtained the reduction of a rectangular figure of two unequal sides into a square of equal content.

§ 89. Such a proportion may evidently be continued by successive addition or subtraction of antecedents or consequents to a series of either increasing or decreasing numbers, as well as an arithmetical one; producing quantities having a common factor between each step, which is called the *common index* or *constant ratio*; and the progression, or series of quantities resulting from it is called a *geometrical progression* or *series.* In the increasing progression the common index is a whole number, and in the decreasing one it is evidently a fraction; it corresponds likewise, as in the ratio itself, to the quotient arising from the division of two successive terms.

The following is an example of such a progression or series:

$$64 : 32 = 32 : 16 = 16 : 8 = 8 : 4 = 4 : 2 = 2 : 1 = 1 : \tfrac{1}{2} = \tfrac{1}{2} : \tfrac{1}{4} = \tfrac{1}{4} : \tfrac{1}{8} = \tfrac{1}{8} : \tfrac{1}{16}.$$

This is also usually written omitting the signs of equality, and the terms are separated by the sign of addition (+) instead of the sign of division (:), because this notation is better adapted to the use made of these series in higher calculations, where they are of great utility; the above series may then be written thus:

$$*S = 64 + 32 + 16 + 8 + 4 + 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \text{\&c.}$$

Every subsequent number being here the half of the preceding one, the common index of the series is $= \frac{1}{2}$; or any one of the numbers multiplied by $\frac{1}{2}$ will produce the number immediately succeeding it.

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* S, being considered as designating the value of the series.

Thence in a geometrical series the third term is the third proportional to the first two, the fourth is a fourth proportional, the fifth a fifth proportional, and so every subsequent term presents that proportion to the first two terms which its number in the geometrical series indicates.

It is proper here to drop this subject for the present in order to take it up in a later part of the work, when we shall investigate its consequences and practical applications.

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CHAPTER IV.

Rule of Three.

§ 90. In the preceding chapter we have found: that a geometrical proportion is the same with the equality of two fractions, and that *the products of its extreme and mean terms are equal.* We proceeded in the demonstration thus: the numerator and denominator of the two equal fractions were multiplied *each by the denominator of the other;* equal denominators being obtained by it, the conclusion was that the numerators were also equal.

If, instead of multiplying both factors by the denominators mutually, we multiply only one in numerator and denominator, the equality will evidently remain, because the value of the fraction so multiplied does not change. Thus we obtain from the proportion

$12 : 3 = 16 : 4$ or, expressed as a fraction, $\frac{12}{3} = \frac{16}{4}$

by multiplying the first fraction in numerator and denominator, by the denominator of the second,

$$\frac{12 \times 4}{3 \times 4} = \frac{16}{4}$$

by operating equally upon the second fraction,

$$\frac{12}{3} = \frac{3 \times 16}{3 \times 4}$$

In both cases the two fractions having one of the factors in the denominator equal, the same principle applies to this equal factor, as to the equal whole denominators before, according to what is known of the principles of fractions; they therefore compensate each other in this equality, and we obtain

by the first: $$\frac{12 \times 4}{3} = 16$$

and by the second: $$12 = \frac{16 \times 3}{4}$$

That is: we obtain one of the terms expressed by the three others; and this in such a manner, that the product of either *extremes* or *means* being made, and this divided by the one *mean* or *extreme*, the result gives the other *mean* or *extreme*.

As we have seen; that the mutations allowed in geometrical proportion admit any one term to be made either extreme or either mean, under the corresponding mutations of the other terms, we shall generally, by dividing any one of the products by one factor of the other, obtain a result equal to the other factor of that product.

Thus we would deduce from the above proportion all the following results; viz:

$$\frac{12 \times 4}{3} = 16; \quad \frac{16 \times 3}{4} = 12; \quad \frac{12 \times 4}{16} = 3; \quad \frac{16 \times 3}{12} = 4$$

This is the complete *principle and mode of performing* what is called *the rule of three;* from the circumstance that *three quantities*, or numbers, *are used to determine a fourth*.

If, therefore, any *ratio* between two known quantities is said to be the same as (or equal to) the ratio between one other known quantity and an unknown one, the above principle gives the determination of this unknown quantity by the above process, adapted to the given case or question; and any of the mutations shown in the preceding chapter can be applied to it, as may be required.

§ 91. We will now, authorised by the foregoing proofs, make the application of the principles of *geometric proportion* to the *practical operations* of the *Rule of three.* As it will often be necessary to act upon the unknown quantity as if we knew it, in order to make such of the above demonstrated mutations as may be required, we shall here introduce the method so advantageously practised in universal arithmetic, namely, to denote the unknown quantity by a letter, and choose for that, always, one of the last letters of the alphabet, as x, y, &c.; and when we shall have this letter alone on one side of the sign of equality, we have seen from what has already been said, that the unknown quantity is determined by the combinations or the arithmetical operations indicated of the known ones presented on the other side of this sign of equality; that is, the number obtained by them will be the value of this unknown quantity. All this is but a small extension of the use of signs to denote the operations of arithmetic, which has been introduced in the very beginning, and found so useful in expressing distinctly the operations of arithmetic.

Though it is evidently indifferent in which of the four places of the proportion the unknown quantity stands, a habit prevails, of stating the proportion so that the unknown term occupies the fourth place in the proportion; we shall follow it, wherever the combinations do not present reasons for another arrangement.

1*st Example.* To determine the unknown quantity in the proportion

$$15 : 7 = 19 : x$$

The product of the two mean terms divided by the first extreme will, as proved above, give the value of x, or the other extreme, which is the quantity sought; thus

$$\frac{7 \times 19}{15} = x = \frac{133}{15} = 8 + \frac{13}{15} = 8,8666 + \&c.$$

which, placed in common examples, as has been fully shown in multiplication and division, stands thus:

$$\begin{array}{r}19\\7\\\hline\end{array}$$

$$\begin{array}{ll}\text{product} = & \dfrac{133}{15} = 8 + \dfrac{13}{15} = \text{result.}\\ \text{divisor} & \\ & \ \ 13\end{array}$$

or by continuing the division into decimal fractions:

$$\frac{133}{15} = 8,8666 + \&c.$$

$$\begin{array}{l}130\\\ \ 100\\\ \ \ \ 100\\\ \ \ \ \ \ 10\end{array}$$

when the division continued would evidently give a continued succession of the 6.

Thus therefore, the fourth term, or x, is determined; and any other proportion, or rule of three, the terms of which are ever so great or complicated, may be solved by the same operations, performed upon the respective numbers.

2nd Example. Suppose that 7 men mow 37 acres of meadow in a certain time; how many acres will 27 men mow in the same time?

Here we have given: the *ratio between the men* employed, to which, by the nature of the subject, the ratio between the acres of meadow, mowed by each number of men respectively, must be equal; of this only the number of acres mowed by the 7 men is given, and the number of acres that can be mowed by 27 men is the quantity sought, which we have agreed to designate first by a letter, as x.

If, therefore, we make the number of men corresponding to the number of acres given, the first antecedent term of the *geometric proportion*, the second number of men will be the first consequent or second term of the proportion; the antecedent of the second ratio, that is, that of the number of acres mowed in each case, must be the 37 acres; as corresponding to the work of the number of men forming the antecedent in the first ratio; the number of acres corresponding to the number of men, whose work it is intended to ascertain by the operation, here our x, must therefore be the consequent of the second ratio, or the fourth term of our proportion. This gives therefore the statement:

$$\begin{array}{cccc} \textit{Men.} & \textit{Men.} & \textit{Acres.} & \textit{Acres.} \\ 7 : & 27 = & 37 : & x \end{array}$$

And by the operation shown above, and deduced before, we obtain:

$$x = \frac{27 \times 37}{7} = \overset{\textit{Acres.}}{142} + \frac{5}{7} = \overset{\textit{Acres.}}{142{,}714} \text{ \&c.}$$

where the decimal fractions are evidently carried far enough for any practical purpose in the case.

I have been thus long and detailed in this first example of the application of geometric proportion to the rule of three, to show the details of the reasoning which must guide in the statement of a practical question; that I may be allowed in future to suppose them known, and that I may have to explain only the peculiarities which may occur in other cases, in the same manner as I here suppose the arithmetical operations of multiplication and division as sufficiently explained in the first example.

The scholar will now observe: that in performing the arithmetical operations, the things or objects, which the numbers represent, do not enter into the consideration, and that the numbers alone are treated, as indicative of the relation of these things in regard to quantity, according to our first definition of quantity; for, what would a product of men into acres of land represent in nature? But the division made again by a number representing men, might even be considered as compensating, in a manner similar to that of the equal factors in the numerator and denominator of a fraction, which compensate each other; and there then remains, we might say, the denomination of acres in the numerator, to give the denomination to the result.

This is exactly analogous to what has been said at the beginning of this part of arithmetic; that the *ratio* only of the two things of the same kind is taken, as the principle that determines the ratio of two other things, which may be of a nature completely different from the first two. We shall in general find, in all results of calculations relating to objects of different kinds: that the denomination of the result is that of the kind of quantity, or things, which appear in it an odd number of times, and that those which appear in an even number of times act as mere numbers, giving no denomination to the quantity of the result. This remark, which is here very simple, becomes of great importance in higher calculations, and is in all cases an indispensable property of an accurate result.

3d Example. My neighbour bought 372,45 acres of land for $720,5, but I can dispose of only $215,5 for that purpose; how much land can I purchase at the same rate?

The ratio of the money is here given, and the ratio of the land purchased by it must of course be the same; we have therefrom the statement:

$$\$720{,}5 : \$215{,}5 = 372{,}45 \text{ acres} : x \text{ acres.}$$

This proportion can be reduced to simpler numbers, by dividing corresponding terms by 5, which is a common factor; it is therefore proper to do it; thus it becomes:

$$144{,}1 : 43{,}1 = 372{,}45 : x$$

which gives $$x = \frac{43{,}1 \times 372{,}45}{144{,}1} = 111{,}399 \text{ acres.}$$

Here it was evidently most proper to proceed altogether by decimal fractions, in which also the answer fits best.

4th Example. If 57 lb. 7 oz. of spices be bought for $17,25, what must I pay for 87 lb. 10 oz. 7 dwt.?

Here the ratio of the spices is given, and the quantities contain denominate fractions; we would have to divide the second by the first, which is, as shown above, a very inconvenient operation; we may therefore either reduce the weights to the lowest denomination of the denominate fractions, which is the pennyweight, and then proceed as in whole numbers, or reduce these denominate fractions to decimal fractions of the pounds. We have seen above that the first is the most convenient, when we do not foresee that the denominate fractions will give short and determinate decimals; we shall therefore proceed by this reduction; thus we obtain for the two first numbers,

$$\begin{array}{rcr}
57 & \text{and} & 87 \\
\underline{12} & & \underline{12} \\
114 & & 174 \\
\underline{577} & & 87 \\
691\text{ oz.} & & \underline{10} \\
\underline{20} & & 1054\text{ oz.} \\
13820\text{ dwt.} & & \underline{20} \\
 & & 21080 \\
 & & \underline{7} \\
 & & 21087\text{ dwt.}
\end{array}$$

by multiplying first the pounds by 12, to reduce them to ounces, and adding the ounces given, then multiplying by 20 to reduce to pennyweights, and adding the pennyweights given; thus we obtain the statement:

$$13820 : 21087 = 17{,}25 : x$$

Dividing the antecedents by 5, to reduce:

$$2764 : 21087 = 3{,}45 : x$$

which gives: $x = \dfrac{3{,}45 \times 21087}{2764} = \$20{,}32$

In the decimal fractions resulting we stop at the 32 cents, no mills coming after; further accuracy would be useless.

5th Example. A sum of money being shared between John and James in the proportion of 9 to 4, it results that John has $15 more than James; what were the shares of each? and what was the whole sum shared?

The proportion stated from the above data stands thus:

$$\begin{array}{ccc} & \textit{John's.} & \textit{James's.} \\ 9 : 4 = & x + 15 : & x \end{array}$$

Subtracting the consequents from the antecedents, and comparing with the consequents we obtain:

$$9-4:4=x+15-x:x$$

or $\quad 5:4=15:x$

Dividing by 5, $\quad 1:4=3:x$

which gives $\quad x=3\times 4=12=$ James's share

and $\quad x+15=27=$ John's share

and the whole sum $\quad =39$ has been shared.

6th Example. Two merchants make a joint stock; they contribute in the proportion of 14 to 5; the difference between the full shares is \$504; what was each individual's share and the whole stock?

$$\textit{Ans.}\begin{cases}\textit{Shares} & \begin{cases}\$\ 280\\ \quad 784\end{cases}\\ \textit{Stock} & \quad 1064\end{cases}$$

which is obtained by exactly the same process as above.

7th Example. Three merchants make a joint stock; the first puts in a certain unknown part of the capital, the second 2000 dollars more, and the third 3000 dollars less, than the first; the ratio of the shares of the second and third is as 9 to 5; what are all the individual shares, and the stock itself?

If we call the share of the first, which regulates the whole question, x, we shall have the statement thus:

$$9:5=x+2000:x-3000$$

Comparing the difference between the antecedents and consequents with the consequents, we obtain:

$$9-5:5=x+2000-x+3000:x-3000$$

or $\quad 4:5=5000:x-3000$

Dividing the antecedents by 4; $\quad 1:5=1250:x-3000$

whence $\quad 5\times 1250=x-3000$

$$6250=x-3000$$

That is, the share of the third is $\quad =\$6250$

The share of the first is therefore $\quad =9250$

" " second " $\quad =11250$

And the whole stock $\quad =26750$

8th *Example.* A bankrupt leaves clear property $84421,26; his creditors are as follows; viz:

Jones for	$ 5629
Williams	14207
Rufus	592
King	29768
Eldridge	120352

What dividend in the hundred, or proportional part, can be paid, (under the supposition of equal concourse,) and what will each creditor get for his share?

Here the ratio of the sum of the debts to the clear property will be the constant ratio, which will give the rule for the division; each claim forms the second antecedent, or what is the same thing, the first term of the second ratio. Or, the fraction arising from the division of the property by the sum of the debts, which may be most easily expressed in decimals, will be a constant multiplier for each of the individual debts and the shares will be the product of this fraction by the amount of the claim. Thus

$$\frac{84421,26}{170548} = \frac{42210,63}{85274} = 0,495$$

will be a constant multiplier for each of the claims, which will give the shares as follows:

Of Jones,	$2786,355
Williams,	7032,465
Rufus,	293,04
King,	14735,16
Eldridge,	59574,24

EXAMPLES FOR PRACTICE.

1. If $58\frac{1}{2}$ gallons of wine cost $28,50, what will 620 gallons come to?

2. If 562 yards of linen cost $495, how many yards can be bought for $1051?

3. If upon 52 acres of land 962 bushels of wheat have been harvested, how many bushels would 225 acres yield at the same rate?

4. What will be the amount to pay for 652 lb. 6 oz. of coffee, if for every 57 lb. 6 oz. I must pay $15,15.

5. A man having bought 359 yards of cloth for $621, what must he sell them at to make 15 per cent. upon the sale?

6. A merchant bought 795 yards of cloth for $107,50, he has still $427,50 which he wishes to lay out in the same cloth, at the former price; how many yards may he yet purchase?

7. If the matting for the floor of a room 24 feet by 18 cost $95,60 what will the same matting come to for a room 22 feet in length by 38 in breadth?

8. The forage required by a body of cavalry, for a month of 31 days, is 2821 cwt. of hay, how much will be needed for the same body for 87 days?

9. How many pounds of tea can a man buy for $672, if he buy 751 lbs. for $327,50?

10. If 21 men could perform a work in 17 days, and 16 men be added to them after the second day, how much time will be saved by it?

11. The annual wages of a man being $100, to be paid in land at $6 per acre, how many acres will he receive after 3 years and 7 months?

12. Two men, A and B bought together 200 acres of land, each paying $200; they divide, and A making choice of the better land, they agree to value his land at $2,25 the acre, and that of B at $1,75; how many acres will each of them get?

13. If the interest of money is 7 per cent. what will be the discount?

14. How much must a man pay down to receive in 6½ years $658, the interest being 7 per cent. calculating upon simple interest?

15. On the importation of certain goods, a merchant gains 20 per cent. when the duty is 16½ per cent.; what per cent. will he gain upon the same, when the duty is raised to 18 per cent.?

16. A man had rented a farm of 150 acres for $235; now he is offered another of 225 acres for $380; how much would he gain or lose by accepting the offer, all other circumstances considered equal?

17. If a man has offered to him 135 yards of linen for $115, and another offers him 212 yards of the same qual-

ity for $200, how much is the one offer more advantageous than the other, and which is the more advantageous?

18. If a man travels 19 miles in 6 hours and 5 minutes, how far will he walk between sun-rise and sun-set, when the sun rises at 5 o'clock 25 minutes, and sets at 6 o'clock 35 minutes in the evening, stoppages being included in both cases?

19. The freight of 13 tons of goods to a certain place cost $$29\frac{1}{2}$, what freight will have to be paid at the same rate for $59\frac{1}{2}$ tons?

20. A man obliged to live on his income of $3750 a year was under a rent of $325 a year. By the depreciation of the funds, his income is reduced to $2960, and he is now asked for the same house the rent of $255; can he afford to pay it, if he will put only the same proportion upon his rent as before?

21. A merchant gaining $7500 in 6 years with a capital of $18000, what would he gain at the same rate in 14 years?

§ 92. In many cases in nature, and the common intercourse of life, the things whose ratio is compared, augment, the one in the same ratio as the other diminishes, and inversely; as for instance, the more men are about a work, the less time it will require to do it; the quicker a man walks, in proportion to another man, the less time he will require to go through a certain space; and so in many other cases in nature. That is to say: the *ratios* (of these things) and that of the results are *inversed* in relation to each other. Therefore, in all such cases, the ratio of the two given terms of the same kind is also to be inverted in the statement of the proportion, and then the operation of the rule of three is to be executed with this inverted ratio, in the same manner as above with the direct one; this operation is evidently grounded on the nature of the things, or the question; as in the following examples.

1*st Example.* I have a meadow, which 6 men usually mowed in 17 days; but, the season being precarious, I wish to have it mowed in 3 days; how many men must I employ?

Evidently the shorter the time, the more men I must employ, so the ratio of the men is the inverse of that of the time; and as this latter ratio is given, I must write it inversely; thus the statement becomes:

$$\begin{array}{lc} & \textit{Days.} \quad \textit{Men.} \\ & 3 : 17 = 6 : x \\ \text{or} & 1 : 17 = 2 : x \\ \text{giving} & x = 2 \times 17 = 34 \text{ men;} \end{array}$$

and so many men must be employed to do in 3 days the work of 6 men in 17 days.

2d Example. Two men, starting at the same time, ride a certain distance; John travels at the rate of $6\frac{1}{3}$ miles an hour, and Peter $7\frac{3}{4}$ an hour; Peter arrives after 20 hours 20 minutes; when will John arrive?

The ratio of the time of arrival is evidently the inverse of that of the speed, or number of miles made per hour; therefore the statement must also be inverted; thus:

$$\begin{array}{c} \textit{Miles.} \qquad \textit{H. Min.} \\ 6\frac{1}{3} : 7\frac{3}{4} = 20.\ 20 : x \end{array}$$

Fractions occurring here, they must be reduced; but 20 minutes being a third of an hour, and the fraction $\frac{1}{3}$ occurring in the first term, we may take advantage of it to shorten this operation thus: reducing the whole numbers to fractions upon this consideration, we obtain:

$$\frac{19}{3} : \frac{31}{4} = \frac{61}{3} : x$$

Multiplying by 3 $\quad 19 : \frac{31}{4} = 61 : x$

The fraction of the second term may be left unreduced, and the result written thus:

$$x = \frac{31 \times 61}{4 \times 19} = \frac{1891}{76} = 24{,}88157 \text{ hours.}$$

As 60 minutes make one hour, every tenth of an hour is 6 minutes; the decimals of hours are therefore reduced to minutes by multiplying by 6, and remarking that the result of the tenths gives the units of the minutes, or the denominate fraction of 60 parts, or $\frac{1}{60}$, the above becomes thereby 24 h. 52,8942 m. The same subdivision is continued to the seconds, the same reduction will reduce the decimals of minutes into seconds and decimals of seconds; thus: 24 h. 52 m. 53,652 s. = time of arrival of John.

3*d Example.* In a besieged place the garrison consists of 2000 men; in a retreat, 600 throw themselves into it, to escape the enemy; the provisions of the place were sufficient for the former garrison for 250 days; how long will they last the increased number of men, at the same rate of daily allowance?

Of course the greater the number of men, the less time the provisions will last, and that in the inverse ratio of the original to the augmented garrison; thus we have the statement:

$$\overset{\textit{Men.}}{2000+600} : \overset{\textit{Men.}}{2000} = \overset{\textit{Days.}}{250} : x$$

or $$2600 : 2000 = 250 : x$$

Dividing the first ratio by 2000; $$1,3 : 1 = 250 : x$$

This gives $$x = \frac{250}{1,3} = 192,3 \text{ days,}$$

that is; the provisions will leave a small remainder after 192 days, as we obtain only three tenths of a day over.

4th Example. A father, leaving a property of \$76743, makes the regulation in his will: that it shall be divided between his two sons in the inverse ratio of their ages; the one is $12\frac{1}{2}$ years old, the other 16 and 4 months; what will be the share of each?

In this question the inversion consists only in the condition of the disposition itself, namely: that the age of the one shall determine the share of the other mutually; and the sum of the ages forms the antecedent term of the (comparison or) ratio, given for the proportional share of each in the whole amount; we have therefore, expressing the months as twelfth parts of the year, the following statement:

$12+\frac{6}{12}+16+\frac{4}{12} : 12+\frac{6}{12} = \$76743$: sh. of the older;

$12+\frac{6}{12}+16+\frac{4}{12} : 16+\frac{4}{12} = \$76743$: " younger;

or, by successive reductions of each, which will be easily followed:

$$12 + \tfrac{3}{6} + 16 + \tfrac{2}{6} : 12 + \tfrac{3}{6} = \$76743 : x$$

and $$12 + \tfrac{3}{6} + 16 + \tfrac{2}{6} : 16 + \tfrac{2}{6} = 76743 : y$$

or $$28 + \tfrac{5}{6} : 12 + \tfrac{3}{6} = \text{''} \quad \text{''}$$

and $$28 + \tfrac{5}{6} : 16 + \tfrac{2}{6} = \text{''} \quad \text{''}$$

or $$\tfrac{173}{6} : \tfrac{75}{6} = \text{''} \quad \text{''}$$

and $$\tfrac{173}{6} : \tfrac{98}{6} = \text{''} \quad \text{''}$$

lastly $$173 : 75 = \text{''} \quad \text{''}$$

and $$173 : 98 = \text{''} \quad \text{''}$$

giving the share of the older $$= \frac{75 \times 76743}{173} = \$33270{,}0867$$

" younger $$= \frac{98 \times 76743}{173} = \$43472{,}9133$$

which produce again the full property.

EXAMPLES FOR PRACTICE.

1. If 23 heads of cattle can pasture in a field for 57 days, how long can 17 heads of cattle pasture in it?

2. If a canal could be finished by 350 men in 321 days how many days will it take if only 298 can be got to do the work?

3. How many days longer will 58 gallons of beer last a family that usually consumes 3 quarts a day, if they reduce their allowance to 5 quarts for two days?

4. It takes, to clothe a regiment of 750 men, 5920 yards of yard wide cloth, how many yards of cloth of $1\tfrac{5}{8}$ yards wide, will it require to clothe the same?

CHAPTER V

Compound Rule of Three.

§ 93. From the principle explained in section 87, we derive, as is there stated, the *Compound Rule of Three;* where several proportions being given, which all concur in the determination of an unknown quantity, the product of the different proportions term for term being made, the same principle, of the equality of the products of the extreme and mean terms, takes place, as in simple proportion, and the same arithmetical process gives the means of determining the unknown quantity. It is necessary, of course, to pay proper attention to the nature of the ratios given, in respect to whether they are direct or inverse, and to make the statement of each accordingly.

As the operation in itself has already been explained in section 87, and as we shall immediately explain a simple and general principle, by which all such compound influences and effects as produce a compound proportion, or, what is called the compound rule of three, can be calculated with the greatest ease, whatever may be their complication; we will here only apply it to such examples as have for their first ratio units of different denominations, and form thereby what in mercantile calculations is called the *Chain Rule.* This comprehends the finding of the equivalent of exchange, weight, or measure, of two places, by means of the given ratios of intermediate places, when the direct ratio is not known. This operation will exemplify still more strikingly the remark made above, in relation to the compensations of the denominations in the multiplications and divisions, resulting from the operations of the rule of three.

1*st Example.* If 60 lbs. weight at Paris, make 50 lbs. at Amsterdam, and 45 lbs. at Amsterdam, 50 lbs. in New-York; how many pounds of New-York make 720 lbs. of Paris?

Multiplying these proportions, term for term, we obtain the compound proportion by the products, as below:

$$\begin{array}{l} P.\ \ A. \\ 1 : 1 = 60 : 50 \end{array}$$

$$\begin{array}{l} A.\ \ N.Y. \\ 1 : 1 = 45 : 50 \end{array}$$

$$\begin{array}{l} N.Y.\ \ P. \\ 1 : 1 = x : 720 \end{array}$$

$$\begin{array}{l} P.\ \ A.\ \ N.Y.\ \ A.\ \ N.Y.\ \ P. \\ 1 \times 1 \times 1 : 1 \times 1 \times 1 = 60 \times 45 \times x : 50 \times 50 \times 720 \end{array}$$

$$1 : 1 = x : \frac{50 \times 50 \times 720}{60 \times 45}$$

$$1 : 1 = x : \frac{2000}{3}$$

or $$x = 666{,}666$$

by equality of products of extremes and means.

The products of the unities of the first ratios, give the ratio of unity to the product of the second ratios; the denominations in the first ratios are all compensated, as observed before, and we obtain, by dividing in the second compound ratio by the numbers multiplying the x, the result of this equality, which is then only reduced as a fraction.

2nd Example. A merchant of Petersburg has to pay in Berlin 1000 ducats, which he wishes to pay in rubles by the way of Holland; and he has for the data of his operation, the following proportional values of moneys, viz: that 1 ruble gives 47,5 stivers; 20 stivers make 1 florin; 2,5 florins make 1 rix dollar Hollandish; 100 rix dollars, Hollandish fetch 142 rix dollars Prussian; and finally, 1 ducat in Berlin is 3 rix dollars Prussian; how many rubles must he pay? This gives the following statement:

$$\begin{array}{lllll}
1 \textit{ ruble} & : 1 \textit{ st.} & = 47{,}5 & : 1 \\
1 \textit{ st.} & : 1 \textit{ fl.} & = 1 & : 20 \\
1 \textit{ fl.} & : 1 \textit{ r.d. H.} & = 1 & : 2{,}5 \\
1 \textit{ r.d. H.} & : 1 \textit{ r.d. Pr.} & = 142 & : 100 \\
1 \textit{ r.d. Pr.} & : 1 \textit{ duc.} & = 1 & : 3 \\
1 \textit{ duc.} & : 1 \textit{ ruble} & = x & : 1000
\end{array}$$

By the same process as in the former example, is obtained:

$$x = \frac{1000 \times 3 \times 100 \times 2{,}5 \times 20}{47{,}5 \times 142} = 2223{,}87 \textit{ rubles.}$$

§ 94. In the activity which nature presents to us, as well as in all our actions, we observe this principle: that *the product of any cause into the time of its action is equal to the* *effect of it.* Or, the product of any means whatsoever, into the time of their action, or the power which acts upon them, or the conventional law of their action, produces a determined effect; that is, it is equal to it. Thus we have seen, that a capital loaned on interest, renders as the product of the rate of interest into the time; that a man's labour, is the result of the product of his strength (or power) into the time he exercises this strength (or power.) In all this therefore, we see nothing but the simple multiplication of certain factors, and their product; as has been quoted in the remarks to section 73. In the same manner as products in arithmetic may be the result of a continued multiplication, so may an effect in nature be the combined product of a number of causes, means, powers, or times; and the effect itself may be represented by a combined product; as occurs, for instance, in higher mechanics, where these quantities often appear as multiplied by themselves, or in the square, cube, &c.

§ 95. If we now consider the relation of two such effects; that is to say, their ratio to each other, we find, as we have done in simple numbers, that: *the same ratio must take place between the Products of cause into time,* (as it will be simplest to call that by a general name,) *as that existing between the effects.*

We have now for some time made use of letters to denote quantities, before we knew the numbers which would correspond to them; we shall here extend the advantage derived from it, in order to present this idea at one glance in its full connexions, and with the arithmetical operations connected with it. For that purpose we shall designate the objects of calculation, or the quantities of them, by their initial letters, and call

$$\left.\begin{array}{ll}\text{the cause} & = C\\ \text{the time} & = T\\ \text{the effect} & = E\end{array}\right\}\text{for one of the objects;}$$

and for the other, which is compared to it in the compound proportion, we shall call the same objects by the corresponding small letters, as:

the cause $= c$
the time $= t$
the effect $= e$

We then obtain, by the principles stated already in the remark to section 73:

$$C \times T = E; \quad \text{and} \quad c \times t = e$$

and for the proportion arising from this, in a manner exactly similar to what has been done in common numbers, we obtain the statement:

$$C \times T : c \times t = E : e$$

which corresponds, as simple products expressed by their factors, and their results, to a statement similar to

$$\begin{array}{ccccccccccc} C. & & T. & & c. & & t. & & E. & & e. \\ 3 & \times & 4 & : & 7 & \times & 9 & = & 12 & : & 63 \end{array}$$

It evidently follows from this, by the division of the corresponding terms of the proportion, that we have also:

$$C : c = \frac{E}{T} : \frac{e}{t} \quad \text{and} \quad T : t = \frac{E}{C} : \frac{e}{c}$$

This says in words: *the Ratio of the Causes is the same as that of the Effects divided by the Times*; and: *the Ratio of the Times is the same as that of the Effects divided by the Causes.*

The numbers of the example will thence give

$$3 : 7 = \frac{12}{4} : \frac{63}{9} \quad \text{and} \quad 4 : 9 = \frac{12}{3} : \frac{63}{7} \; *$$

§ 96. As we have seen in the preceding application of geometric proportion to the rule of three, that whatever term of the proportion be unknown, if the three others are given, this fourth is determined by the principles of the proportion; so in the present case, whatever may be the quantity unknown in such a compound rule of three, whether a cause, a time, or an effect, or a part of the one or the other of them, this quantity will be determined by the others, and obtained by the appropriate mutations of the proportion, or the operations of arithmetic resulting from it.

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* The teacher who will take the trouble to speak with his scholar upon this principle, or the attentive reader, who will compare it with the circumstances that surround him, will have no difficulty in explaining this idea; its correctness and generality will prove a great facility to the intelligent arithmetician. My own experience has proved to me that it meets no difficulty with boys of about 12 or 14 years, as scholars usually are, when in common schools they are thus far advanced in arithmetic, and that they made the statements appropriated to it very readily, and with peculiar satisfaction. It furnishes the best exercise of the mind for the appropriate application of common arithmetic. The examples which follow are worked out, and will, I hope, lead the way to its proper and easy application.

By this consideration and process all the complication, often resulting from combinations of direct and inverse proportions, in a compound rule of three, which are apt to lead young calculators into mistakes, are avoided, because every quantity, in any way concerned, is by its nature placed as factor in its proper place, by the simple reflection of its acting as either cause, time, or effect.

It may be easily seen that it will solve with ease questions upon combined actions of capitals during different times, as well in interest, as in shares of profit or loss, that is, in partnership, in complicated questions upon combined works, and all similar cases, as the following examples will show.

1st Example. A capital of \$6200 produces in 5 years, at 7 per cent. \$2170, amount of interest; what will a capital of \$9300, at 4 per cent. produce in 9 years?

Here the statement is extremely simple, thus:

$$\overbrace{6200 \times 0,07}^{C} \overset{T}{\times 5} : \overbrace{9300 \times 0,04}^{c} \overset{t}{\times 9} = \overset{E}{2170} : \overset{e}{x}$$

This proportion may evidently be much reduced; first, by dividing by 100, it becomes,

$$62 \times 0,07 \times 5 : 93 \times 0,04 \times 9 = 2170 : x$$

Dividing the first ratio by 2,

$$31 \times 0,07 \times 5 : 93 \times 0,02 \times 9 = 2170 : x$$

Dividing the antecedents by 70,

$$31 \times 0,001 \times 5 : 93 \times 0,02 \times 9 = 31 : x$$

Dividing the antecedents by 31,

$$0,001 \times 5 : 93 \times 0,02 \times 9 = 1 : x.$$

The second antecedent being reduced to unity, this gives

$$x = \frac{93 \times 0{,}02 \times 9}{5 \times 0{,}001} = \frac{16{,}74}{0{,}005} = \$3348$$

That is, the capital of $9300, at 4 per cent. produces, in 9 years, $3348 interest?

2nd Example. A capital of $9500, at 6 per cent. interest, annually, produced $4560 in 8 years, at what rate of interest must a capital of $12000 be lent out, which shall render $4800 in 5 years?

$$\overbrace{C \qquad Y}^{C} \qquad T : \overbrace{c \qquad i}^{c} \qquad t = E : e$$

$$9500 \times 0{,}06 \times 8 : 12000 \times x \times 5 = 4560 : 4800$$

Reducing as above, by dividing the first ratio by 500, and the second by 40,

$$19 \times 0{,}06 \times 8 : 24 \times 5 \times x = 114 : 120$$

Dividing the two antecedents by 6,

$$19 \times 0{,}01 \times 8 : 24 \times 5 \times x = 19 : 120$$

Dividing the two antecedents by 19, and the consequents by 24,

$$0{,}01 \times 8 : x \times 5 = 1 : 5$$

Dividing the two consequents by 5, and executing the multiplication indicated in the first term, we obtain:

$$0{,}08 : x = 1 : 1$$

Or, the rate per cent. $= x = 0{,}08$ or 8 per cent.

Thus the simple reductions of the proportion given, has furnished the result. It is evident that if we had at the first outset of this and the preceding example, expressed the term in which x is, by the other three, we would have reached the same results by the compensations in the numerator and denominator, and the factors of x with the opposite numerator, but the principles appear clearer under the form used.

3d Example. Two men in partnership, contribute as follows: A puts in $7521 which he withdraws after five years and a half. B puts in $9772, which act in the company during 6 years, before which time the accounts cannot be settled. It is required to determine the share of each in the general result of all the operations, (which are taken together,) amounting to a net profit of $15472?

The sum of the products of the stocks into the times of their acting, are here to be compared to each single product of stock into the time of its acting, as cause and time; the whole benefit evidently represents the effect, corresponding to the whole stock, and its time of action.

Thus we obtain the two following statements:

$$7521\times5{,}5+9772\times6 : 7521\times5{,}5 = 15472 : \text{share of A}$$

$$7521\times5{,}5+9772\times6 : 9772\times6 = 15472 : \text{share of B}$$

$$\text{Or } 99997{,}5 : 41365{,}5 = 15472 : \text{share of A}$$

$$\text{And } 99997{,}5 : 58632{,}0 = 15472 : \text{share of B}$$

Here we evidently obtain, as in the case of a bankrupt, treated in a former example, a constant fraction from the third term divided by the first, with which the second, or the product of the stock into the time of each partner is to be multiplied, to obtain his share in the profit; or we have:

$$\text{The share of A} = \frac{15472}{99997{,}5}\times 41365{,}5 = 6400{,}2$$

$$\text{The share of B} = \frac{15472}{99997{,}5}\times 58632 = 9071{,}8$$

The fractional part being reduced to decimals gives 0,1547238, which being multiplied into the whole number 58632, gives the result indicated.

4th Example. If 180 men, working 6 days, each day 10 hours, can dig a trench of 200 yards long, by 3 yards wide, and 2 yards deep, how many days will 100 men take to dig a trench of 360 yards long, 4 wide, and 3 deep, by working 8 hours in a day?

This gives the following statement, in which the effect is a compound product, because the trench has the three dimensions of length, breadth, and depth. The reductions which it admits, will here be made without mentioning them, under the supposition that the preceding examples have shown the principle of them; y being taken for the unknown days.

$$180 \times 10 \times 6 : 100 \times 8 \times y = 200 \times 3 \times 2 : 360 \times 4 \times 3$$

$$18 \times 6 : 8 \times y = 10 : 36$$

$$9 \times 3 : 2 \times y = 1 : 3{,}6$$

$$27 : y = 1 : 1{,}8$$

$$y = 27 \times 1{,}8 = 48{,}6 \text{ days.}$$

5*th Example.* A hare is 50 leaps before a greyhound, and he takes 4 leaps while the greyhound takes 3; but 2 greyhound's leaps are equal to 3 hare's leaps; how many leaps must the greyhound make to overtake the hare?

This, as it appears a standing question in all books on arithmetic, is well adapted for an example in this case

The proportion of the leaps as given, are:

In time; hare's leap : hound's leap = 4 : 3
In length; " : " = 2 : 3

The compound ratio of them, or the product of cause into time, which determines the effect, is therefore:

$$\text{hare} : \text{hound} = 8 : 9$$

If we call the distance the hound has to run $= x$, in hare's leaps, (as the determined distance is given in this kind of quantity,) the hare's run will be $x - 50$ in the time they both run; these two circumstances of the data give the following statement:

$$x : x - 50 = 9 : 8$$

By comparing the antecedent with the difference between antecedent and consequent, we obtain:

$$x : 50 = 9 : 1$$

$$x = 9 \times 50 = 450 \text{ hare's leaps}$$

As the hare's leaps are $\frac{2}{3}$ of the hound's this distance will require 300 hound's leaps; so many therefore, he will have to make to overtake the hare.

6*th Example.* If 9 men, working 6 days, at the rate of 8 hours per day, can build a wall of 152 feet long, and 9,5 feet high, how many days must 16 men work, at the rate of 10 hours each day to build a wall 295 feet long, and 17,5 feet high?

EXAMPLES FOR PRACTICE.

1. If 352 men, having worked 8 hours every day, have made a certain length of canal in 87 working days, and there remains now $\frac{1}{7}$ of the same length to be done to complete the work, which it is intended they should do in 8 working days; how many hours more per day must they work to complete the task at the same rate of working?

2. If 6 men pave 55 yards of a street in 5 days, how many men will it take to pave 212 yards in 12 days?

3. A man performing a journey in 21 days by walking 7 hours at the rate of 5 miles an hour, how many days will it take him to perform the same journey by walking 10 hours at the rate of $3\frac{1}{2}$ miles an hour?

4. Of Brandy sold at $1,25 per gallon, there is bought for the amount of $87, and of brandy worth $1,35, for the amount of $96; a mixture of one third of the first quantity and two thirds of the second quantity being made, what will be the proportional price of it if it is required to clear 10 per cent. upon the sale?

5. If 248 men, in 5 days, working 11 hours a day, dig a trench 280 yards long, 3 wide and 2 deep; in how many days of 9 hours each, will 32 men dig a trench 430 yards long, 6 wide, and 3 deep.

6. If 5 men, in 10 days, mow 42 acres of meadow, how much will 13 men mow in 18 days?

7. If 23 boards, of $12\frac{1}{2}$ feet long and 14 inches broad, make a certain flooring, how many boards will it take of 15 feet long and 10 inches broad?

8. A merchant gaining $6750 in 4 years, with a capital of $15000, what would he gain at the same rate in 7 years with a capital of $32000?

9. If 172 boards, 17 feet 6 inches long and 14 inches broad, are needed to floor a place, how many would it take 12 feet 6 inches long and 10 inches broad?

10. If 2100 bushels of oats feed 200 horses during 21 days, at $\frac{1}{2}$ a bushel per day, how long will 3700 bushels last 760 horses, at $\frac{5}{8}$ of a bushel per day?

11. How many yards of paper, 22 inches broad, will cover a wall of 26 yards circuit and 9 feet high, if 20 yards circuit of the same height can be covered by 72 yards of 30 inch wide paper?

12. What provision must be made for an army of 9560 men, in bread, if they shall receive 2 lbs. per day for 70 days; if found by experience that 5000 men will need in 25 days, 312500 lbs. at the rations of $2\frac{1}{2}$ lbs. per day?

13. The common step of a horse being about 4 feet, and that of a man $2\frac{2}{3}$ feet, the man making 8 steps to the horse's 5, how much space will the man gain over the horse, in walking a distance of 18 miles?

* * *

CHAPTER IV.

General Application of Geometric Proportion.

§ 97. When two proportions are given, two unknown quantities may be determined by means of the mutations of these proportions, and the determination of the one by the three others; appropriating the choice of the operations to the given case, in such a manner that, by whatever operation the quantity sought is involved with other given quantities, these become disengaged by performing the contrary operation; this is grounded upon the principle of arithmetic stated in the beginning, that each operation (or rule of arithmetic) has its opposite operation; and this is the principle used in all the reductions that have been made in the proportions in the preceding sections, to obtain, or render easy the obtaining of the results, as it is also that used in all higher calculations, in algebra, &c.

1*st Example.* Two numbers are in the ratio of 2 : 3; when each is augmented by 4, they are in the ratio of 5 : 7; what are these numbers?

Denoting the one by x, the other by y, we have the first statement:

$$2 : 3 = x : y$$

And as the fourth term is equal to the product of the two mean terms divided by the first, we have also:

$$2 : 3 = x : \frac{3 \times x}{2}$$

that is, $$y = \frac{3 \times x}{2}$$

The second proportion, by using this result, will be stated thus:

$$5 : 7 = x + 4 : \frac{3 \times x}{2} + 4$$

Multiplying the second ratio by 2;

$$5 : 7 = 2x + 8 : 3 \times x + 8$$

By subtracting antecedents from consequents:

$$5 : 2 = 2x + 8 : x$$

Subtracting twice the consequents from antecedents;

$$1 : 2 = 8 : x$$

whereby $$x = 2 \times 8 = 16$$

And y by the first proportion, placing the value of x, just found, in its place:

$$2 : 3 = 16 : y$$

or $$1 : 3 = 8 : y$$

whence $$y = 3 \times 8 = 24$$

2nd Example. A father being asked how many sons and daughters he had, answered, "If I had two more of each, I should have three sons to two daughters, and if I had two less of each, I should have two sons to one daughter;" how many sons and how many daughters had he?

This evidently furnishes two proportions, one stated by the sums of the numbers sought and 2; and the other by the difference between the numbers sought and 2, as follows.

Calling the number of the sons $= x$;

That of the daughters $= y$;

$$x + 2 : y + 2 = 3 : 2$$

$$x - 2 : y - 2 = 2 : 1$$

From these proportions are obtained, by steps grounded upon the principles of proportion, demonstrated in section 86, the following successive results:

From $$x+2 : y+2 = 3 : 2$$

$$x+2 : x+2-y-2 = 3 : 1$$

$$x+2 : x-y = 3 : 1$$

In like manner, from

$$x-2 : y-2 = 2 : 1$$

$$x-2 : x-2-y+2 = 2 : 1$$

$$x-2 : x-y = 2 : 1$$

Dividing these two results term by term, as by section 86:

$$\frac{x+2}{x-2} : 1 = \frac{3}{2} : 1$$

or $$x+2 : x-2 = 3 : 2$$

From this $$x+2+x-2 : x+2-x+2 = 5 : 1$$

or $$2x : 4 = 5 : 1$$

and $$x : 2 = 5 : 1$$

$x = 2 \times 5 = 10$ the number of sons.

Though this determines the number of daughters, if we place this value in either one of the first proportions, and then determine the y, as in the foregoing example; still it is evident that both x, and y, are dependent upon the data in exactly the same manner; I will therefore also determine y by a similar appropriate process, as it will be a good example to show the principles of this use of proportions in determining quantities in general.

We made the first term, containing x, our standing term; we shall have now to make the second term, containing y, the standing term of the operation. Thus we have from the first proportion:

$$x+2-y-2 : y+2 = 1 : 2$$

or $$x-y : y+2 = 1 : 2$$

And from the second proportion;

$$x - 2 - y + 2 : y - 2 = 1 : 1$$

$$x - y : y - 2 = 1 : 1$$

Dividing these two proportions term by term, as before, we obtain:

$$1 : \frac{y + 2}{y - 2} = 1 : \frac{2}{1}$$

or $$y - 2 : y + 2 = 1 : 2$$

By sum and difference:

$$y + 2 + y - 2 : y + 2 - y + 2 = 3 : 1$$

or $$2\,y : 4 = 3 : 1$$

$$y : 2 = 3 : 1$$

giving $y = 2 \times 3 = 6$ for the number of daughters.

3d *Example.* I asked my two neighbours, John and Peter, how many heads of cattle each had; Peter, thinking to puzzle me, says, "Our cattle, taken together, are to what John has more than I, in the ratio of 3 to 2; and if we multiply the two numbers of our cattle together, that product will be to all our cattle in the ratio of 5 to 3." I find how many each of them has in the following way:

Calling John's cattle $= x$

and Peter's cattle $= y$

the first proportion given furnishes me the statement,

$$x + y : x - y = 3 : 2$$

and the second,

$$x + y : x\,.\,y = 3 : 5$$

By addition and subtraction of the first proportion is obtained:

$$x + y + x - y : x + y - x + y = 5 : 1$$

or $$2\,x : 2\,y = 5 : 1$$

$$x : y = 5 : 1$$

thence $x + y : y = 6 : 1$

and $x + y : x = 6 : 5$

Dividing the second proportion given by either of these, term for term, I get:

$$\frac{x+y}{x+y} : \frac{x \cdot y}{y} = \frac{3}{6} : 5 = \frac{1}{2} : 5$$

and $$\frac{x+y}{x+y} : \frac{x \cdot y}{x} = \frac{3}{6} : 1 = \frac{1}{2} : 1$$

that is, $1 : x = 3 : 30 = 1 : 10$

and $1 : y = 1 : 2$

giving $x = 10 ;\quad y = 2$

So I find John has 10 heads of cattle, and Peter appears to be richer in puzzles than in cattle, which he did not like to tell me.

4th Example. A, B, and C, in a joint speculation, gain, and give only the following account of the quantity each gained: the product of the gain of A into that of B is equal to $1200, that of A into that of C = $1800, and that of B into C = $2400; what was the gain of each?

This example will show, that *an equality of products* as is given here, *expresses a geometric proportion equally as well as an equality of fractions or ratios*; for by the decomposition of these products into the extreme and mean terms of a proportion, we obtain the three proportions:

$$x : 40 = 30 : y$$

$$x : 60 = 30 : z$$

$$y : 40 = 60 : z$$

Dividing the first by the second, term by term, we obtain:

$$\frac{x}{x} : \frac{40}{60} = \frac{30}{30} : \frac{y}{z}$$

or $60 : 40 = z : y$

Dividing this proportion by the third, term for term:

$$\frac{60}{y} : \frac{40}{40} = \frac{z}{60} : \frac{y}{z}$$

$$60 : y = z \times z : 60y$$

$$60 : z = z : 60$$

$$z = 60$$

Dividing the first and third, term by term:

$$\frac{x}{y} : \frac{40}{40} = \frac{30}{60} : \frac{y}{z}$$

$$x : y = 30z : 60y$$

$$x : 1 = z : 2$$

Multiplying this by the second, term for term:

$$x \times x : 60 = 30 \times z : 2z$$

$$x : 30 = 30 : x$$

$$x = 30$$

Dividing the second by the third, term for term:

$$\frac{x}{y} : \frac{60}{40} = \frac{30}{60} : \frac{z}{z}$$

$$40x : 60y = 30 : 60$$

$$4x : y = 3 : 1$$

$$x : y = 3 : 4$$

Dividing this by the first, term for term:

$$\frac{x}{x} : \frac{y}{40} = \frac{3}{30} : \frac{4}{y} = \frac{1}{10} : \frac{4}{y}$$

or $$40 : y = y : 40$$

$$y = 40$$

This example, expressly chosen for its simplicity, may suffice to explain the principle.

EXAMPLES AND QUESTIONS.

1. Two travellers, A and B, leave two places, 100 miles distant from each other, at the same time; A travels $6\frac{1}{2}$

miles per hour, and B $7\frac{3}{4}$ miles per hour, what part of the distance will each of them make?

Ans. $\left\{\begin{array}{l} A = 44,9205. \\ B = 55,0295. \end{array}\right.$

And what time will they travel before they meet?

Ans. 7 *h.* 6 *min.* nearly.

2. How many yards of cloth were there in a piece which cost $66,60, the price of the yard being to the number of yards, as 5 to 7? *Ans.* 9,5561.

3. The sum of two numbers multiplied by the greater is 120, the same multiplied by the less is 105, what are the 2 numbers? *Ans.* 8 *and* 7.

4. The slow, or parade step of the military being 90 steps per minute, and the step 28 inches, how far would troops travel, by marching 8 hours in a day?

5. The hour and minute hand of a clock are together at 12 o'clock, when are they together after each hour afterwards?

6. Of two travellers upon the same road, A travels 5 miles an hour, B 3 miles an hour; when B passes a certain place on the way, A is still 13 miles behind him; at what distance will he overtake B? *Ans.* $32\frac{1}{2}$ *miles.*

7. Two men bought a lottery ticket in partnership, A gave $9 towards it, B gave $7; the ticket draws a prize of $2000, how much will each of them get?

Ans. $\left\{\begin{array}{l} A = 1125. \\ B = 875. \end{array}\right.$

8. The father of a child is 52 years older than the child, his mother 36 years older, and the age of the father is to that of the mother as 4 to 3; what is the age of the child? *Ans.* 12 *years.*

9. The product of the sum of two numbers by the greater, is equal to 209, and by their difference, equal to 57, what are these numbers? *Ans.* 8 *and* 11.

10. The sum of two numbers multiplied by the greater, gives 24 times the lesser, and multiplied by the lesser, gives 6 times the greater; what are the two numbers?

11. Three workmen can severally do a piece of work in the following times: A in 3 weeks; B in 8 weeks may

perform it 8 times; C 5 times in 12 weeks; in what time will they perform the work jointly?

Ans. in $\frac{8}{9}$ of a week.

12. If A and B together can perform a work in 8 days, and A and C in 9 days, and B and C in 10 days; how many days will it take each to perform the work alone?

Ans. To A = $14\frac{34}{49}$; to B = $17\frac{23}{41}$; to C = $23\frac{7}{31}$.

13. The sides of two squares are in the ratio of 3 to 1, and the sum of their surfaces in feet, is to the sum of their sides in the ratio of 15 to 1; what are the sides of the squares? *Ans.* 18 *ft. and* 6 *ft.*

14. A, B, and C make a joint stock; A put in $170 less than B, and $340 less than C, and the sum of the shares of A and B, is to the sum of the shares of B and C, as 6 to 7; what did each put in?

Ans. share of A = 425; of B = 595; of C = 765.

15. A, B, and C increase a certain stock they have, in equal shares, so that A adding $3000, B $5000, and C $8000, these 3 shares are in continued geometric proportion, what was the share of each in the original stock, and what are the new shares of each?

$$\text{The original stock was \$1000, the new sh.} \begin{cases} 4000 \text{ of A.} \\ 6000 \text{ of B.} \\ 9000 \text{ of C.} \end{cases}$$

16. The product of two numbers is 63, and the square of their sum is to the square of their difference, as 64 is to 1? what are the numbers? *Ans.* 9 *and* 7.

17. The difference of two numbers when divided by the lesser, is equal to 48 divided by the greater; and when divided by the greater, is equal to 3 divided by the lesser; what are the numbers? *Ans.* 4 *and* 16.

18. The sum of two numbers when divided by the greater, is equal to $\frac{7}{5}$, and when multiplied by the lesser, is equal to 126; what are these two numbers?

Ans. 6 *and* 15.

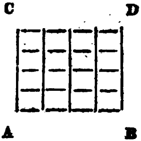
PART IV.

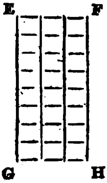
EXTENSION OF ARITHMETIC TO HIGHER BRANCHES AND OTHER PRACTICAL APPLICATIONS.

CHAPTER I.

Of Square and Cube Roots.

§ 98. When in a multiplication the two factors are equal, the product is called a *square*; because it corresponds to what would be produced in nature by laying off the quantity which these numbers represent, in any unit of lineal measure, in two directions perpendicular to each other; and completing the figure by two equal lines, drawn perpendicular at the end of these; as, for instance, taking 4 feet and laying them off upon *AB*, and also upon *AC*, and then drawing *BD*, and *CD*, at equal distances, again perpendicular to *AB*, and *CD*; *ABCD* will be a square, representing the square of 4, that is, $4 \times 4 = 16$.





The product of any two numbers may be represented in the same way, by two lines perpendicular to each other, divided into equal parts, and completing the rectangular figure, having its opposite sides equal; as here the figure *EFGH*.

So we may, when we have such a surface, or product, given, and one of the sides, find the other side by division, as is evident from the second figure. But when the figure is a square, as in the first case, we can find the two equal sides of it by a peculiar process, which is called the *extraction of the square root*; the principle of which it is now intended to explain.

For this purpose it is necessary to investigate what a product is composed of, by decomposing each factor into two parts, not unlike the method we have used to show the propriety of the principle of carrying in multiplication; namely, we divide the number into two parts; thus, for instance, we would write 14 as $10 + 4$; or merely consider it as so composed, and by multiplying the number into itself under that form, keeping each individual result separate, we shall obtain the following process and results.:

$$\begin{array}{r}
14 \\
14 \\
\hline
4 \times 4 \\
4 \times 10 \\
10 \times 10 + 4 \times 10 \\
\hline
10 \times 10 + 2 \times 4 \times 10 + 4 \times 4 \\
100 + 80 + 16 = 196
\end{array}$$

That is, we obtain by the product of the units $4 \times 4 = 16$; by the product of the unit of the multiplier into the tens of the multiplicand, $4 \times 10 = 40$, and the same again by the product of the tens of the multiplier into the units of the multiplicand; then lastly, by the product of the tens, $10 \times 10 = 100$.

This gives, by the addition, three distinct products, viz:

1*st*. The square of the first part, that is, the product of the first part into itself, here 10×10.

2*nd*. Twice the product of the two parts into each other, here twice 4×10, or $2 \times 4 \times 10$.

3*d*. The square of the last part, or the units, here $= 4 \times 4$.

In making the division of the number according to our decimal system of numeration, they follow the same order in magnitude as here stated. We find also by the inspection of this result, as we know besides by the multiplication table, that the product of the units can influence two places of figures, namely, units and tens, and cannot influence the third; the same is the case with any of the subsequent numbers, each influencing only the rank which it occupies, and the next higher rank; this gives the principle, by which we may know in any number, of how many numbers the square root will be composed, namely: by dividing it into as many pairs of figures, from the right hand side towards the left, as it will admit; the number of these divisions, will be the number of figures of the *square root.*

As the *extraction of the square root* of a number will again be the opposite of the elevation to the square, the above operation must be executed in an inverted order to extract the square root, as in division the inverse order of the multiplication has been followed.

The operation of raising to a power is also called, *Involution*, and the extracting of the root, *Evolution.*

In order to denote in an abridged manner the multiple of a number by itself, the idea will readily occur, to write the number only once, and to indicate the number of factors intended, by placing a small number at the top and to the right hand of the number, corresponding with this number of factors; and $10^2 = 10 \times 10$; $10^3 = 10 \times 10 \times 10$; and so for any other. To indicate the extraction of the root the sign $\surd$, or an extended *r*, is written before the number; as $\sqrt{196}$, denotes the square root of 196; if other roots are to be extracted, the number corresponding to the degree of the root is written in the $\surd$, as $\sqrt[3]{\ }$; $\sqrt[4]{\ }$; and so on; but a much better method is, to continue the same manner of notation as in raising numbers to their powers, expressing the roots in their corresponding fractions, so that $\sqrt{196} = (196)^{\frac{1}{2}}$; $\sqrt[3]{196} = (196)^{\frac{1}{3}}$; and so on in higher degrees.

§ 99. In Evolution the first step will therefore be, as in division, to find that number which, multiplied into itself, will give the product nearest below the most left hand subdivision of the given number, whether this consist of one or two numbers; this square being subtracted, the remainder must furnish the two other products; as the second of these is the larger, if we multiply the number found before by 2, and divide the remainder of the given number by it, we shall have a number as quotient, near the second, or next following number; with which we shall then have to execute the two products, indicated by the above result of such a multiplication. But in the determination of this quotient it must be observed: that there must be left a sufficient latitude for the subtraction of not only its double product with the first number found, but also of the square of this quotient; as is evident from the result obtained by the raising of a number into the square.

1*st Example*. Let the above number be chosen to extract the square root, to explain the direct inversion of th operation; or to execute

$$\begin{array}{rl}
 & \sqrt{1|96} = 10 + 4 = 14 \\
\text{First square } 10 \times 10 = & 1|00 \\
\text{Remainder} = & 96 \\
\text{Divisor } 2 \times 10 = & 20)\ \text{in } 96;\ 4 \text{ times} \\
(20 + 4) \times 4 = & 96 \\
\text{No remainder} & 00
\end{array}$$

The number divided off by 2 from the right hand shows that the root has two places of figures; so the first will be in the tens, and the number in the second division being 1, the square root of which is also 1, the first square will be 10 × 10 = 100; the root therefore is 10; this being written, the square = 100, is subtracted from the whole 196; the remainder, 96, being written, the divisor, which shall serve to find the other number, will be the product 2 × 10 = 20; which being found to go 4 times in the remainder, 4 is written in the root, and being also added to the 20, the sum of both is multiplied by 4 again, as we have found it to be factor in both the two last terms; the product of this = 96, written under the remainder 96, being exactly equal, gives the 14 as the square root of 196 in return.

2*nd Example*. Let it be given to extract the square root of a number of more than two places of figures, as 13456. Dividing the number off as before directed, we find, that the root must have three places of figures, or the first figure will be in the hundreds; thus we obtain the following process;

	√ 1	34	56	= 100 + 10 + 6 = 116
100 × 100	1	00	00	
	—	—	—	
Divisor 2 × 100		34	56	quotient = 10
Product 2×100 + 10 multiplied by 10 =		21	00	
		—	—	
Remainder =		13	56	
Divisor = 2 × 110				quotient = 6
(220 + 6)×6 =		13	56	
		—	—	
		0	00	

It will easily be conceived, that every number does not give a whole number for a radical, because every number is not the product of another number multiplied into itself; in the same manner as in division every number is not divisible by a given number.

We may evidently in large numbers, by way of abridgment, take only the two next numbers down, as in division, and consider the former as a ten, in relation to this number taken down, and proceed thus to the end, or to any desired number of places of decimals; for the process, as first mentioned, will proceed in this case according to the same system, exactly as if it was a mere division continued to decimals, only the mode of making up the successive products which are to be subtracted being different. Therefore, also, the evolution of a number with decimal fractions is exactly the same as the evolution of whole numbers, whether it have an exact root or not. But it must be remarked, what the principles upon which decimal fractions are grounded might easily suggest, namely: that the partitioning into pairs must begin from the unit and proceed equally, both to the left and to the right; therefore, if there be an odd number of decimal places, a 0 must be placed to the right to make up the pair, which, as is well known, does not change the value of the fraction.

The following two examples will suffice to give a correct idea of it, and lead to the practice of this operation.

To execute √ 1419,7864, being a number with a decimal fraction: writing it partitioned off as just directed, the following results:

	√ 14	19,	78	24	= 37,68
First square	9				
Remainder =	5	19			quot. = 7
Divisor = 2 × 30					
(60 + 7)7 =	4	69,			
Remainder =		50,	78		
Divisor = 2 × 37,0					quot. = 0,6
(74,0 + 0,6)0,6 =		44,	76		
Remainder =		6,	02	24	
Divisor = 2 × 37,6					quot. = 0,08
(75,20 + 0,08)0,08 =		6,	02	24	
		0	00	00	

Here the process is evident, from the expressions placed opposite to each number; the number obtained is always augmented by a 0, and multiplied by 2, to form the divisor, which from the remainder gives the next figure; this is considering it as the ten of the following number; the quotient added, and the sum multiplied by it, gives the product to be subtracted; and the remainder is to be treated as before.

Exactly in the same manner the following example gives √ 2; it is here placed without any further indication, in order to give room for study

√ 2		= 1,41421356 + &c.							
1									
1	00								
	96								
	4	00							
	2	81							
	1	19	00						
	1	12	96						
		6	04	00					
		5	65	64					
			38	36	00				
			28	28	41				
			10	07	59	00			
			8	48	52	69			
			1	59	06	31	00		
			1	41	42	13	25		
				17	64	17	75	00	&c.

EXAMPLES.

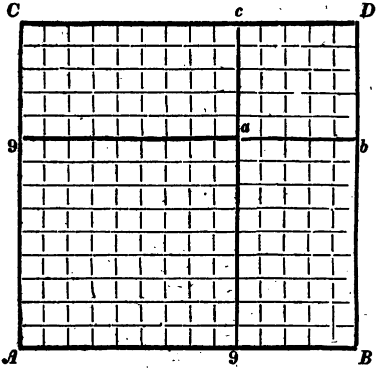
Extract the square root of

1*st.*	1296	8*th.*	7,
2*nd.*	7921	9*th.*	18,49
3*d.*	9899	10*th.*	365
4*th.*	25,1001	11*th.*	106920
5*th.*	6905,61	12*th.*	152399025
6*th.*	476991,	13*th.*	0,006
7*th.*	3,	14*th.*	78,5

§ 100. From the preceding we have only a short and easy step to make, by means of reflections grounded upon the principles just used to explain the extraction of the square roots, in order to determine the principles upon which a *Quadratic Equation* is solved; that is, to furnish the means to determine an unknown quantity, which, in a combination with others, would be multiplied into itself, or that, as we have stated above, is said to be *squared.* To make the explanation more simple, we may use two means, which taken in conjunction will, I hope, leave to the attentive student of this book, no difficulty.

I wish to introduce this here, although unusual, because its absence would leave us in the subsequent parts, when we shall treat of progressions, without the means of finding, or satisfactorily explaining, the solution of certain questions arising from them; for I have proposed to myself, never to lead the student blind over any step; while at the same time I wish to give him all the means of calculation in arithmetic, that he may desire, in a manner satisfactory to a reflecting mind.

We have before decomposed the number, of which we wished to show the different products forming the square, into two parts, and have there shown, that the square number resulting was composed of the sum of the squares of the two parts, and twice the product of the two factors into each other; we there decomposed the 14 into 10 and 4; we choose this division on account of its direct application to the extraction of the square root of a number written in our usual decimal system; but any division will do the same thing.



If in the annexed figure, of 14 subdivisions on each side, we divide the sides into 9 and 5 parts, the result will be exactly the same; we shall have the square $A9a9 = 9 \times 9, = 81$; the product of 5×9 twice, on each side of this square, in $9abB$, and $9\ acC$, and the small square $abDc = 5 \times 5$ which together will fill up the large square $ABDC$; and summing up these products, obtained by the multiplication as above, we obtain

$$9 \times 9 + 2 \times 9 \times 5 + 5 \times 5 = 81 + 90 + 25 = 196.$$

and any other division would give the same ultimate result.

As, therefore a square number can be decomposed in any two parts, so as to obtain from it two smaller squares, and twice the product of the two parts into each other, we are allowed to consider any square number to be thus composed.

We have seen in the very beginning, that in arithmetic we have always two operations, exactly opposite to each other, the one always compensating the effect of the other. We have seen, in treating of proportions, that when the same operation was executed on both sides of the sign of equality, the results were again equal, and therefore the principle of equality still subsisted; or, what is the same, that equal operations performed upon equal quantities do not destroy the equality; by this means we were enabled to obtain solutions of questions, or, what is the same, determine unknown quantities, variously involved by other known ones.

If now, in application of these principles, we consider an unknown quantity in any manner involved, which appears in any one or more of the parts, multiplied into itself, that is, in the square; and in other parts simple; we are, by the principle last shown, authorised and enabled to separate the square from all other numbers, or quantities that might multiply or divide it; and we can consider it thus insulated, according to the explained principles of the division of the square, as representing the square of the first part, or subdivision of the number, or part of the whole square.

To apply this to an example, we must again give our unknown quantity a designation, and treat it as if we knew it, until it comes to stand alone on one side of the sign of equality, which gives the solution, by indicating that it is equal to the result of the combination represented by the known quantities on the other side of the sign of equality. Then the terms multiplied by the unknown quantity must be considered as representing twice the product of the first term into the second, or, in that case, of the unknown quantity into the known ones. The known or determined part of this will therefore represent the double of the second part, therefore half of this factor being squared will represent the smaller square; (or in general the other square needed to complete the entire square;) therefore by the addition of this square on both sides of the equality, a square number is obtained, of which the square root can be extracted by the rules given, or, what is in this case equivalent, which can be expressed by the given numbers. The quantity sought for is therefore known from it.

Example. Suppose we had given, by the result of a calculation, a combination of quantities which have the following form:

$$\begin{aligned}
480 &= 3x^2 + 36x \\
160 &= x^2 + 12x \\
196 &= x^2 + 12x + 36 \\
\sqrt{196} &= x + 6 \\
14 - 6 &= 8 = x
\end{aligned}$$

having the x in the square multiplied by 3; this must first be disengaged, by dividing all the terms on both sides by 3; this gives the second line; then the 12, multiplying the simple x, represents the product of 2 into the second part of the subdivision of the whole square; therefore its half, or 6, is the side of this second square, when x is the side of the other, because the $12x$, or $2 \times 6 \times x$, must represent the double product of the two parts, like $9\,abB + 9\,acC$. If, therefore, we square the 6, and add it to both sides, by which the principle of the equality is retained, we shall have on the right hand side a full square, in which the x is the side of one of the lesser squares, and the other is known; thus, the third line above is obtained; the two parts, into which the square appears divided, are therefore x and 6, which will together be equal to the square root of 196; this gives the fourth line. Extracting the square root of 196, gives 14, and if the 6 is subtracted on both sides, the difference gives the value of x as in the last line, for the final result.

The operations needed in consequence of the above principles are therefore the following.

1. *Write the given quantities in such an order, that the parts containing the unknown quantity stand all on one side of the sign of equality; and those having none but known quantities on the other side.*

2. *Arrange it so that the square of the unknown quantity multiplies at once all the quantities which it has to multiply, and do the same with the quantities that multiply the unknown quantity simply. These multiples may be one or more quantities and either whole or fractional.*

3. *Disengage the square of the unknown quantity of all its multipliers, either whole or fractional, by dividing every term of the equation by them.*

4. *Make the square of the half of the factors which multiply the unknown quantity in the simple form, and add this square to both sides.*

5. *Extract the square root of that side of the equation which has no unknown quantity, and write on the side of the unknown quantity the root of this unknown quantity and of the square added.*

6. *Subtract the known part which now appears added to the side of the unknown quantity from the square root of the determined number of the other side.*

7. *The result will be the value of the unknown quantity sought.*

These general principles will include all cases that may occur.

EXAMPLES IN QUADRATIC EQUATIONS.

1. Given $x^2 - 8x - 7 = 13$ to find x. *Ans.* 10.
2. " $3\,.\,x^2 - 2x = 40$ " x. *Ans.* 4.
3. " $\frac{1}{2}x^2 - \frac{1}{3}x + \frac{11}{2} = 9$ " x. *Ans.* 3.
4. " $3x^2 + 2x - 9 = 76$ " x. *Ans.* 5.
5. " $2x^2 - 54x = 56$ " x. *Ans.* 1.

6. To divide ten into two parts, so that their product shall be equal to twelve times their difference.
Ans. 4 *and* 6.

7. To divide 13 into three parts, so that the difference between the squares shall be equal, and the sum of the squares $= 75$. *Ans.* 1, 5, *and* 7.

§ 101. For the *cube*, or the product of three equal factors, which corresponds in nature to the solid, we have to multiply the product, which has been obtained for the square, once more by the first quantity; in order to show what different parts it is composed of, the above mode of separating the factors is to be preserved, because it will show how the products are to be made in the extraction of the cube root. For this purpose, the same example which has served before will be again made use of.

We have obtained in section 98, by 14 × 14, or 14^2, the result

$10\times10+2\times10\times4+4\times4$ which being multiplied

by $10+4$ gives

$$
\begin{array}{l}
\hline
10\times10\times10+2\times10\times4\times10+4\times4\times10 \\
\qquad\qquad + 4\times10\times10 \quad +2\times4\times4\times10+4\times4\times4 \\
\hline
10\times10\times10+3\times10\times10\times4+3\times10\times4\times4+4\times4\times4
\end{array}
$$

$$= 10^3+3\times10^2\times4+3\times10\times4^2+4^3 = 2744 = 14^3$$

It will be observed, that this product is composed of the cube of 10; three times the square of 10 into 4; three times the product of 10 into the square of 4; and the cube of 4. Or, generally, the cube of the first part, and three times the product of the square of the first part into the second; then three times the product of the first into the square of the second part; and lastly, the cube of the second part.

These products are therefore to be formed out of the parts of a cube the root of which it is intended to extract.

It will again be observed here, that, with reference to the subdivision of the cube in the order of our decimal system, the second term will be the largest after the first, as it contains the double square of the first, as the largest factor which may occur after the cube of the first; it forms, therefore, the leading part, or factor, to find the second part, as in the extraction of the square root.

It will also appear, that, as we had to divide off the number into pairs of figures in the square, here it will be necessary to divide off the number every three figures, from the right hand side towards the left, because the product of a number of two figures into one of one figure may give three figures in the result.

With these results, and the principles which arise from them, for the converse operation, that is, the extraction of the cube root, we shall be able to execute this operation properly.

1*st Example.* The above resulting number, 2744, being given, to extract the cube root, which is indicated thus:

	$\sqrt[3]{2}$	744	$= 14$
First cubic root taking off 1^3	1		
Remainder	1	744	
Divisor $= 3 \times 10 \times 10 =$		300	quot. $= 4$
Second term $= 300 \times 4 =$	1	200	
Third term $= 3 \times 10 \times 4 \times 4 =$		480	
Fourth term $= 4 \times 4 \times 4 =$		64	
Sum of the three terms $=$	1	744	
Subtracted from the remainder leaves		000	

The only number which cubed will not exceed 2 is 1; taking away this cube gives the remainder 1744; forming the triple product of the $10^2 = 300$; this in common division would go 5 times in 1744; but there must here be room for the subtraction of the products indicated above, and it will be found that only 4 will admit that; thereby we form the 3 terms placed under, as indicated, according to principles resulting from the formation of the cube; the sum of which is equal to the former remainder, and subtracted leaves 0, giving 14, the exact cube root of 2744.

2nd Example. Extract the cube root of 994011992, or execute

	$\sqrt[3]{994}$	011	992	$=900+90+8=998$
$900^3 =$	729	000	000	
First remainder =	265	011	992	
Divisor = 3×900^2 =	2	430	000	quotient = 90
$3\times 900^2\times 90$ =	218	700	000	
$3\times 900\times 90^2$ =	21	870	000	
90^3 =		729	000	
Sum of factors =	241	299	000	
Second remainder =	23	712	992	
Divisor $=3\times(990)^2$ =	2	940	300	quotient = 8
$3\times 990^2\times 8$ =	23	522	400	
$3\times 990\times 8^2$ =		190	080	
8^3 =			512	
Sum of factors =	23	712	992	
Third remainder =	00	00	00	

The given number admitting three subdivisions in beginning from the right, indicates a root of three places of figures. The nearest cube root of the first division of the numbers on the left being 9, which in the third place is equivalent to 900, the cube being made and subtracted, leaves the first remainder; the triple product of the square of it, taken as a divisor, shows 90 as a quotient, for the next root. The products are now formed as indicated; their sum being subtracted from the first remainder, leaves the second remainder, upon which the same process takes place as before, taking the whole of the root found as the first term; and the sum of the products being equal to the last remainder, the number given proves an exact cube of the number 998 obtained as root.

3d Example. If the number is no exact cube, we may extract the approximate root in decimal fractions, as well as in the square root; the number of 0's to be added each time must of course be three, and the products are formed as required in the former example; the process will go on, in other respects, as has been seen in the square root. To make this strikingly apparent, we will here execute $\sqrt[3]{2}$; thus:

	$\sqrt[3]{2}$					= 1,2599 + &c
=	1					
Remainder =	1	000				adding three 0's
Divisor 3×10^2 =		300				quotient = 2
$3\times10^2\times2$ =		600				
$3\times10\times2^2$ =		120				
2^3 =		8				
Sum of factors =		728				
First remainder =		272	000			adding three 0's
Divisor 3×120^2 =		43	200			quot. = 5
$3\times120^2\times5$ =		216	000			
$3\times120\times5^2$ =		9	000			
5^3 =			125			
Sum of factors =		225	125			
Second remainder =		46	875	000		adding three 0's
Divisor 3×125^2 =		4	687	500		quot. = 9
$3\times1250^2\times9$ =		42	187	500		
$3\times1250\times9^2$ =			303	750		
9^3 =				729		
Sum of factors =		42	491	979		
Third remainder =		4	383	021	000	adding 3 0's
Divisor 3×12590^2 =			475	524	300	quot. = 9
$3\times12590^2\times9$ =		4	279	718	700	
$3\times12590\times9^2$ =			3	059	370	
9^3 =					729	
Sum of factors =		4	282	778	799	
Fourth remainder =			100	242	201	

Adding three 0's, it would be continued as before.

The place of the decimal mark is evidently again determined by the usual principle, namely: where it becomes necessary to add 0's to continue the operation. It is here only marked in the root and in the process omitted as understood that the value of the figures is determined by the well known law of the decimal system.

EXAMPLES FOR PRACTICE.

Extract the cube root of

1st.	9261	*7th.*	1520,875
2nd.	1906,624	*8th.*	216,000
3d.	20570824	*9th.*	5832,761
4th.	4052,24	*10th.*	64,372
5th.	43243551	*11th.*	389017
6th.	103161,709	*12th.*	9092727

§ 102. We here see again, that the principles deduced may lead to the solution of equations of the third degree, as this is called in higher calculations, or to determine a quantity which appears as formed of three equal factors multiplied into each other; together with others involved with the lower powers, that is, the square, and the single power, together with some terms consisting of known quantities; but it is not the province of arithmetic to go into this inquiry; because it requires operations, and produces cases, which are reserved to be solved only in universal arithmetic or algebra.

It is evidently possible to produce the involutions of higher degrees in the same manner that has here been shown for the square and the cube; but the evolution presents increasing difficulties as we proceed, the possible combinations of different factors to the same ultimate result being evidently always more numerous, and therefore, also, the possible roots. Even in algebra there is not yet a general method found to solve such questions, and it steps entirely out of the limits of arithmetic to treat any thing relating to this subject.

CHAPTER II.

Of Progressions, or Series.

§ 103. In mentioning (sections 84 and 89) continued proportions, and the progressions or series which result from their continuance, we referred to a future extension of the subject to the progressions or series, which are intended as the subject of the present chapter.

According as the continued proportion is either an *arithmetical* or a *geometrical proportion,* we obtain by its extension to a greater number of quantities: either an *arithmetical* or a *geometrical progression,* or *series;* each of which has peculiar laws; we shall here begin with the first.

§ 104. *A series of numbers which progresses* increasing, or decreasing, *by the same constant difference,* forms a continued arithmetical proportion, or an *Arithmetical Series.*

This principle is therefore the element of all investigation in relation to the properties of this kind of series; according to it we shall be able to write all the terms successively, and therefore obtain the law of the mutual dependence of all the quantities concerned in it; such a series (which we will call equal to S) will, for instance, be the following:

$$S=2+(2+3)+(2+2\times3)+(2+3\times3)+(2+4\times3)$$
$$+(2+5\times3)+\&c.$$

In the writing of these series the terms are joined by the sign +, which may equally serve to express the arithmetical proportion, as I stated at first, and the constant equality of the difference will become equally apparent by the subtraction of each term from its immediately subsequent term, which gives here the constant difference, 3.

Considering the successive dependence of these terms upon each other, and comparing their value in relation to their distance from the first term, we observe that the constant difference makes its first appearance in the second term, and being afterwards found added in each subsequent term, it will in any term whatever be once less than the number of terms indicates, whether the series be increasing or decreasing. Thus we find it here in the sixth term added five times to the first term. This gives us the principle by which to determine any term, when the first term and the constant difference are given.

It will be of the greatest advantage in the extension of arithmetic in this state of forwardness, to apply the use of letters to denote certain quantities, until they are determined, that we may express our ideas clearly, fully, and briefly, by applying to these letters the signs of arithmetic which have been taught in the beginning. We will therefore generally denote the quantities concerned in our present investigation by proper letters; thus:

Let the first term be designated by, or be $= a$
" constant difference $= d$
" number of terms of the series $= n$
" sum of the series $= S$

Thus we shall be able to express the property, which we have just found, of the value of any term, which we denote by n, by

$$\text{term } (n) = a + (n - 1)\,d$$

And the whole series extended to the term n, would be written thus, (omitting the intermediate terms:)

$$\begin{array}{l} \quad\text{1st}\qquad\text{2nd}\qquad\qquad\qquad (n-1)\text{st}\qquad\qquad\qquad n\text{th} \\ S = a + (a+d) \ldots . (a+(n-2)d) + (a+(n-1)d) + \&c. \end{array}$$

Considering the nth term, it is evident that if, of the three quantities concerned in it, and the whole value of the term itself, any three are given, the fourth may be determined from them, just as we determined the fourth term in a geometrical proportion, notwithstanding that the law of their mutual dependence is very different.

Example. In the above series we had $a = 2$; $d = 3$; let n denote the sixth term. We shall, by putting the values of the letters in their places, and performing the operations indicated, obtain the following:

$$\text{Value of the } 6th \text{ term} = 2 + 5 \times 3 = 17$$

In a similar manner any other term would be obtained, as:

$$\text{The } 21st \text{ term} = 2 + 20 \times 3 = 62; \text{ and so on.}$$

If we had 62 as the value of the term given, and the first term, together with the constant difference, we would evidently obtain the number corresponding to the term, by subtracting the first term from the sum, and dividing the remainder by the difference, then adding a unit to the quotient thus:

$62 - 2 = 60$; then $\frac{60}{3} = 20$. Adding 1, gives for $n = 21$.

In like manner any other part can be found, by reversing the operations accordingly.

If for instance the 10*th* term was given $= 29$, the common difference $= 3$. The first term would be found by subtracting 9 times the common difference from the amount of this 10*th* term, as

$$29 - 3 \times 9 = 29 - 27 = 2$$

If the first term $= 2$, is given, and the number of terms $= 12$, together with the value of that term $= 35$, the 1*st* term being subtracted leaves 33, to divide by one less than the number of terms, that is, 11; which will give 3 for the common difference.

§ 105. The most frequent use of these series, and therefore the principal object of inquiry, is the determination of their sum, by means of the three other quantities concerned in it. The principle of this determination is deduced from the nature of the series, in the following manner:

As we found in arithmetical proportion that the sum of the extremes is equal to the sum of the means, so it is evident that here the sum of the extremes is equal to the sum of any two terms equally distant from them, for the sum of every such *pair of terms* must contain the first term twice, and the constant difference an equal number of times, because these increase in numbers equally from the beginning onward, as they decrease from the end backward.

In the above series we obtain:

By the first and last or 6*th* term:

$$2 + 2 + 5 \times 3 = 19$$

By the second and last but one, or 5*th* term:

$$2 + 3 + 2 + 4 \times 3 = 19$$

By the 3*d* and 4*th* term:

$$2 + 2 \times 3 + 2 + 3 \times 3 = 19$$

And generally, by the 1*st* and *n*th term, we would obtain, adopting the expressions above used, the general value of any pair of terms:

$$a + a + (n - 1)\, d$$

Summing up all these pairs of terms, we would of course obtain the sum of the whole series. But there are as many pairs of terms as the number of terms divided by 2; therefore we may obtain the value of the whole series at once, by multiplying the value found above by half the number of terms; that is, in the above numbers:

$$(2 + 2 + 5 \times 3)\, \tfrac{6}{2} = 57$$

And in the general expression in letters, or, as this is usually called, *Equation:*

$$S = \frac{n}{2}\, (2\, a + (n - 1)\, d)$$

In this general expression again there are only four quantities concerned, three of which being given the fourth is determined; by making such operations upon the above equation as will bring the quantity to be determined alone on one side of the sign of equality, as in this case the S.

§ 106. To determine any quantity in any way involved in such an expression as the above, which in general arithmetic is called an equation, the same principle is made use of as has been shown in proportion, namely, that all such mutations are allowed as do not change the principle, that after the change made, the quantities on each side of the sign of equality are again equal. This leads directly to the consequence, that we are allowed to perform any operation of arithmetic we may wish, upon such an equation, provided we do the same on both sides.

As we have seen above, that the operations, commonly called rules of arithmetic, are of such a nature, that two are always opposite to each other, that is to say, the one will always evolve what the other has involved, or disengage what the other has engaged, we shall naturally in an operation such as is proposed, always perform upon such an equation successively all the operations which will disengage the quantity from all others, until it ultimately be found alone on one side of the sign of equality.

We will therefore now apply these principles to the *equation* before us, to obtain successively expressions, or equations, for each of the quantities concerned, by means of all the others, and so collect the solutions of all the questions upon this subject in a series of Problems.

1st Problem. To find the first term of the series, knowing all the other parts, we would proceed thus:

Taking the original equation

$$S = (2a + (n - 1)d)\frac{n}{2}$$

we will divide on each side by $\frac{n}{2}$; which will disengage this multiplication, and give:

$$\frac{2S}{n} = 2a + (n - 1)d$$

Then, in order to disengage the addition on the right hand side, we will subtract on each side what is added there, to the part containing the first term; this changes the equation thus:

$$\frac{2S}{n} - (n - 1)d = 2a$$

The a, or first term, will now be alone, and therefore be determined, if we divide on each side by 2; this gives ultimately:

$$\frac{S}{n} - (n - 1)\frac{d}{2} = a$$

This, expressed in words, which is in fact a less convenient way than the above expression, which speaks to the eye at once, would be thus: the first term is equal to the difference between the sum of the terms divided by the number of terms, and the product of half the common difference into the number of terms, less one.

Suppose we had the sum of the series: $S = 164$
" " common difference: $d = 5$
" " number of terms: $n = 8$

the above expression would present us the following result:

$$a = \frac{164}{8} - \frac{7\times5}{2} = \frac{164}{8} - \frac{140}{8} = \frac{24}{8} = 3$$

2nd Problem. To find the common difference, we would transform the equation after the first step, thus:

having $$\frac{2\,S}{n} = 2\,a + (n-1)\,d$$

we subtract $2\,a$ on each side, which gives:

$$\frac{2\,S}{n} - 2\,a = (n-1)\,d$$

This divided by by $n-1$, on both sides, gives the result:

$$\frac{2\,S}{n\,(n-1)} - \frac{2\,a}{n-1} = d$$

This expression can be made more convenient for calculation, by subtracting the fractions after reduction to a common denominator. Thus it becomes:

$$d = \frac{2\,S - 2\,na}{n\,(n-1)}$$

And by making the 2, a common multiplier to both terms of the numerator:

$$d = \frac{2\,(S-na)}{n\,(n-1)}$$

Assuming for the letters the values given to them above, we obtain:

$$d = \frac{2\,(164-8\times3)}{8\times7} = \frac{2\times140}{56} = \frac{280}{56} = 5$$

3d Problem. Any two terms, the first being one of them, and the common difference being given, to find the number of terms.

When the first term is subtracted from the other term given, we have the product of the common difference into the number of terms less one as remainder; dividing this therefore by the common difference, we have the number of the term, when we add one to this quotient; as for example:

The first term being 5; the other term given 69; the common difference 4:

Subtracting the first term gives $69 - 5 = 64$;

Dividing this by 4, we obtain $= 16$; to which adding 1, gives the number of the term $= 17$.

4th Problem. To find the distance which two terms in an arithmetical series are from each other, the common difference being given:

If we subtract the two terms from each other, we evidently have for the remainder the product of the common difference into the difference between the terms; therefore, when we divide this remainder by the common difference, we obtain the number expressing the distance of the terms; as for example:

Having the two terms 69 and 92, and the common difference 4, we obtain $97 - 69 = 28$; dividing by 4, the distance of the terms becomes $= 7$.

These problems may evidently be varied in different ways; and I now allow myself the supposition that the scholar will be able to do it by himself, as he may wish or need it.

5th Problem. The sum of the series, the first term, and the constant difference, being given, to find the number of terms.

This solution will lead us into a quadratic equation, the principles of which have been explained above, with the express view to their application in this chapter. It is proper to treat it in the general form; we shall therefore take the first formula, or the equation, for the sum of the whole series, and from it solve the value of n, by the following successive steps:

$$\text{Original equation,} \quad S = \frac{n}{2}(2a + (n - 1)d)$$

Multiplying all by 2:

$$2\,S = n\,(2\,a + (n - 1)\,d)$$

Executing the multiplication by n, indicated, and also that by d, in its place:

$$2\,S = 2\,an + dn^2 - nd$$

Arranging the parts on the right by the powers of n, and making n, a common factor to its multipliers in the first and last term on the right:

$$2\,S = dn^2 + (2\,a - d)\,n$$

Dividing by d to make the n^2 free of factors:

$$\frac{2\,S}{d} = n^2 + \frac{2\,a - d}{d}\,.\,n$$

The $\frac{2\,a - d}{d}$ evidently represents here the double of the second term, which we found above in a quadratic equation; taking then the half of it, squaring it and adding it on both sides, gives:

$$\frac{2\,S}{d} + \left(\frac{2\,a - d}{2\,d}\right)^2 = n^2 + \frac{2\,a - d}{d}\,n + \left(\frac{2\,a - d}{2\,d}\right)^2$$

The square root can now be extracted on the right side, it being an exact square; there being on one side none but known quantities, thus:

$$\sqrt{\left(\frac{2\,.\,S}{d} + \left(\frac{2\,a - d}{2\,d}\right)^2\right)} = n + \frac{2\,a - d}{2\,d}$$

And we have now the quantity which we intend to determine simply added to a known one, which being subtracted ultimately on both sides, will leave us, n, alone, that is, fully determined.

We will now, by way of explanation in numbers, apply this to the numerical series supposed in the first problem above, by placing for each letter (except the unknown, n) its value.

$$\sqrt{\left(\frac{2\times164}{5}+\left(\frac{2\times3-5}{2\times5}\right)^2\right)}=n+\frac{2\times3-5}{2\times5}$$

$$\text{or}\quad\sqrt{\left(\frac{328}{5}+\left(\frac{1}{10}\right)^2\right)}=n+\frac{1}{10}$$

Bringing the parts of which the root is to be extracted under one single number, by the following operations successively:

$$\frac{328}{5}+\frac{1}{100}=\frac{328\times20+1}{100}=\frac{6561}{100}=65{,}61$$

the above will give us:

$$\sqrt{65{,}61}=n+\tfrac{1}{10}=8{,}1$$

$$\text{and}\qquad n=8{,}1-0{,}1=8$$

§ 107. We have seen in section 88, that the continuance of a geometrical proportion produces a series of quantities of which each subsequent *is a product of the preceding one by a constant factor*, either whole or fractional; the first case producing an increasing, and the second a decreasing *Geometric Series* (*or progression*,) which is therefore the constant ratio between the terms, or what we have called the *Index*.

The principles of the geometric series are applicable in all questions that relate to compound interest, annuities, and the like; their principles will here be investigated in a manner similar to that used for the arithmetical series; but upon the principles of the geometric proportion, of which it is the continuance. We will for that purpose proceed by the example of the following series; the sum of which we again call S, to have a point of comparison; the terms are therefore also added, or joined by the sign +.

$$S=3+5\times3+5^2\times3+5^3\times3+5^4\times3+5^5\times3$$
$$+5^6\times3\ \&c.$$

The law of *continued geometric proportion*, that the product of the two extremes is equal to the product of the mean term into itself, evidently holds good here, and we have, for instance, by the product of the first and third term, compared with the second, the following results:

$$3\times5^2\times3 = 5\times3\times5\times3$$

or $$225 = 225$$

And by the same process upon the last term and the second before the last, compared with the one before the last:

$$5^4\times3\times5^6\times3 = 5^5\times3\times5^5\times3$$

or $$2197165625 = 2197165625$$

and also by the first into the last and the second into the one before the last, as

$$3\times3\times5^6 = 3\times5\times3\times5^5$$

or $$140625 = 140625$$

In all cases results evidently identical are obtained.

Comparing the number of the factors of the constant ratio in each term with the number of this term, we find again, as in the arithmetical series: that, as this factor appears of course for the first time in the second term, each term will contain one factor less than the number indicating this term; thus the second term has one factor, the third two, the seventh (as above) six; and in general the nth term will have $n - 1$ factors, exactly in a similar manner as found in the arithmetical series. This consideration enables us to determine any term of the series, for: the nth term of the series above will be $= 3\times5^{(n-1)}$; and if we again adopt general denominations as in arithmetical series, by calling

the first term $= a$
the constant ratio $= r$

we would write the above expression of the nth term $= a \,.\, r^{(n-1)}$; that is, the nth term is equal to the product of the first term into the common ratio elevated to a power one unit less than this number of the term. We may therefore again determine any one of these four quantities when we have the three others given.

§ 108. From the *principles of continued geometric proportion* a formula, or equation, is now to be deduced, expressing the *Sum* of a *geometric series* in general terms. We have seen among the mutations of the *geometric proportion:* that the sum of the two terms of each ratio may be compared with either its antecedent or its consequent; this, applied to continued proportion, where the middle terms are equal, produces the following: applied as example to the first three terms of the above series, namely:

$$3 : 3\times5 = 3\times5 : 3\times5^2$$

whence, by addition:

$$3+3\times5 : 3 = 3\times5+3\times5^2 : 3\times5$$

$$\text{or} \qquad 3+3\times5 : 3\times5+3\times5^2 = 3 : 3\times5$$

by mutating the middle terms.

For reason of the same equality of ratio we can add the next ratio to the antecedents, or by mutation to the two terms of the first ratio, and compare it to any one of the antecedents and its consequents, for which we may take the first and second term and get, for instance, for the next step:

$$3+3\times5+3\times5^2 : 3 = 3\times5+3\times5^2+3\times5^3 : 3\times5$$

$$\text{or, } 3+3\times5+3\times5^2 : 3\times5+3\times5^2+3\times5^3 = 3 : 3\times5$$

Thus we might continue until the first antecedent would contain all the terms except the last, and the first consequent all the terms except the first; the second antecedent being always the first term and the second consequent the second term; or by expressing the sum of all the antecedents by the sum of the whole series less the last term, and the sum of all the consequents by the sum of the series less the first term, we will have a general proportion resulting, expressed in the letters adopted above, and for a series of *n* terms; viz:

$$\left.\begin{array}{c}\text{Sum of an-}\\ \text{tecedents}\end{array}\right\} : \left\{\begin{array}{c}\text{sum of con-}\\ \text{sequents}\end{array}\right\} = \textit{1st}\text{ term} : \textit{2nd}\text{ term.}$$

$$S - ar^{(n-1)} : S - a = a : ar$$

and by subtraction;

$$a - ar^{(n-1)} : S - a = a - ar : ar$$

dividing the antecedents by *a*:

$$1 - r^{(n-1)} : S - a = 1 - r : ar$$

multiplying the antecedents by r:

$$r - r^n : S - a = r(1 - r) : ar = 1 - r : a$$

exchanging the mean terms:

$$r - r^n : 1 - r = S - a : a$$

sum of antecedents and consequents compared with the consequents:

$$r - r^n + 1 - r : 1 - r = S - a + a : a$$

or

$$1 - r^n : 1 - r = S : a$$

which gives:

$$S = \frac{a(1 - r^n)}{1 - r} = a\frac{r^n - 1}{r - 1}$$

The better to impress this operation, and its different steps, I will repeat it here in the numbers of the above series, which will enable us to make the full comparison of its general result with any individual case that may occur. The series chosen gives the following numbers in the first proportion, under the supposition of the number of terms n being 7:

$$\begin{aligned}
S - 3 \times 5^6 : S - 3 &= 3 : 3 \times 5 \\
3 - 3 \times 5^6 : S - 3 &= 3 - 3 \times 5 : 3 \times 5 \\
1 - 5^6 : S - 3 &= 1 - 5 : 3 \times 5 \\
5 - 5^7 : S - 3 &= 5 - 5 \times 5 : 3 \times 5 \\
&= 1 - 5 : 3 \\
5 - 5^7 : 1 - 5 &= S - 3 : 3 \\
5 - 5^7 + 1 - 5 : 1 - 5 &= S - 3 + 3 : 3 \\
1 - 5^7 : 1 - 5 &= S : 3
\end{aligned}$$

$$S = \frac{3(1 - 5^7)}{1 - 5} = \frac{3 - 3 \times 5^7}{1 - 5} = \frac{234372}{4} = 58593$$

Remark. I here permitted the quantity to be subtracted to be the greater, both in the numerator and in the denominator; this, though apparently a contradiction, is compensating on the same ground as has been shown above: that the objects themselves disappear in a rule of three, when they appear equally, both in numerator and in denominator; the result here is therefore equally positive. The signs of addition or subtraction, that is, +, and —, compensate as equal quantities in numerator and denominator, exactly like the quantities themselves. It will easily be seen, that if the series had been a decreasing one, the case would have been the reverse; the ratio being in that case a fraction, the numerator and denominator would both have presented positive numbers, that is, the subtracting quantities, being fractions, would both be smaller than the unit.

The above expression for the value of the sum of a geometric progression is therefore the rule (to express it in the common language of arithmetic) by which this sum is to be calculated. It can be stated very simply thus:

Take the difference between unity and the constant ratio elevated to the power indicated by the number of terms; divide this by the difference between unity and the constant ratio, and multiply the quotient by the first term.

This rule is evidently adapted both to increasing and decreasing geometrical progressions.

§ 109. The foregoing expression, or formula, again presents us four quantities mutually depending upon each other, in the manner expressed by it; we may therefore conclude: that any three of them given, determine the fourth; which might form as many distinct problems, as shown in the arithmetic series; we will here only show how to find the first term, the other parts being given.

The last step of the reduction of the proportion evidently gives:

$$a = S\frac{1 - r}{1 - r^n}$$

or, in words: *Divide the difference between unity and the constant ratio, by the difference between unity and the ratio elevated to the power indicated by the number of terms, and multiply the quotient by the sum of the series.*

To determine the constant ratio, or the number of the term, when the other parts are given, requires more extensive deductions and calculation than the plan of these elements admits of; the first requires a solution of what is called a higher equation, and the second the use of logarithms, which both lie beyond our present limits.

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CHAPTER III.

Of Compound Interest.—Principles of Annuities.

§ 110. We have seen in its proper place, that the calculation of simple interest was a simple multiplication of the capital by the decimal fraction representing the interest per hundred; and in the Compound Rule of Three the other questions have been treated which relate to this subject. But, as well for the transactions of monied institutions, as for various other calculations, in political economy and otherwise, the interest after the year, or any other term agreed upon, is considered as again bearing interest, and thus the interest increases at the same rate as the capital itself. This introduces of course a mode of calculation completely different, and partaking of the nature of the Progressions: its principles shall here be treated separately, and with the addition of payments at determined terms; as the interests or annual payments, called *Annuities*, of which it may be proper here to give only the first principles, without going into the details which more intricate speculations introduce into them, as they would draw us out of our prescribed limits.

We shall take the liberty of making use of letters to designate the quantities, until we give them actual values, by way of example; in order to give to the reasoning that general form which it is so advantageous to introduce in the higher branches of arithmetic. Thus we will call the capital $= C$, and the rate of the per centage $= r$; and proceed with these as if they were known numbers, indicating the operations by means of the signs which we have long been familiar with.

The capital having been one year at interest, it will be worth, together with that interest,

$$C + rC = C\,(1 + r)$$

(for the C multiplies the unit and the rate per cent. $= r$.) This being now the capital on interest for the *second* year, it will produce an interest $= C\,(1 + r)\,r$; and the whole value of the capital and interest at the *beginning* of the *third* year will be the sum of the last year's capital and the interest of the same, namely:

$$C\,(1+r) + C\,(1+r)\,r = C\,(1+r)\,(1+r) = C\,(1+r)^2$$

(for here the $C\,(1+r)$ is again a multiplier for the unit and the rate per cent. $= r$, and so will be the case in each following year.) This capital, at the same interest, in the third year will produce an interest $=$

$$C\,.\,r\,(1+r)^2$$

which added to the last capital, gives at the *beginning* of the *fourth* year the value of

$$C(1+r)^2 + Cr(1+r)^2 = C(1+r)^2(1+r) = C(1+r)^3$$

This is therefore the law of the increase of a capital put out upon compound interest; which for any number of years, say n, would give a definitive sum,

$$S = C\,(1+r)^n$$

or expressing this in words: *In order to obtain the value of the whole capital at the end of the last year, the rate of interest added to unity, raised to the power indicated by the number of years elapsed, is to be multiplied into the original capital.*

To show the same operation in numbers, let us suppose a capital, $C = 7500$, at the rate of 6 per cent. compound interest; this (expressing the per centage in a decimal fraction) evidently gives:

The first year's interest:

$$7500 \times 0{,}06$$

The capital at the end of the first year:

$$7500 + 7500 \times 0{,}06$$

which will be more easily calculated thus:

$$7500 \times 1{,}06$$

The second year's interest will be:

$$7500 \times 1{,}06 \times 0{,}06$$

The capital at the end of the second year:

$$7500 \times 1{,}06 + 7500 \times 1{,}06 \times 0{,}06$$

or, again expressed more simply:

$$7500 \times 1{,}06 \times 1{,}06 = 7500\,(1{,}06)^2$$

It will progress in this manner every year by the power of 1,06; that is, the original capital will be multiplied by 1,06 in continued multiplication of as many factors as the number of years indicates; for instance, at the end of six years we would have:

$$(1{,}06)^6 \times \$\,7500 = 7500 \times 1{,}26247696$$

§ 111. If to the above condition of compound interest we add the condition of annual payments, we have the idea of an *Annuity;* when these payments are supposed larger than the interest, (as in that case the whole might be reduced to simple interest,) it is evident that they must eventually consume the capital itself; and that compound interest must also be allowed upon these payments as well as upon the capital; the conditions of such contracts are therefore varied, and grounded upon various contingencies, and principally upon a combination of chances, particularly the probabilities of life, into which it cannot be our object to enter; the first principle which lies at their root is all that is intended to be shown here. The difference between the capital increased at compound interest, and the payments made, at any time, allowing the same rate of interest, is therefore the value of the annuity at that time; this will be founded upon the following investigation.

We shall here proceed as in the preceding section, calling the annual payment $= p$; and supposing them to begin at the end of the first year: it will afterwards be easy to adapt the result to other conditions of payments, beginning at a later period.

Thus we have,

At the *end* of the *first* year, the amount left

$$= C(1+r) - p$$

At the *end* of the *second* year

$$= C(1+r)^2 - p(1+r) - p$$

At the *end* of the *third* year

$$= C(1+r)^3 - p(1+r)^2 - p(1+r) - p$$

and so on every subsequent year, always deducting from the original capital, with its compound interest at the time, the payments made with their interests, at the same rate, also at compound interest.

So for the *end* of any year, generally named $= n$, we shall have for the amount left, called a, expressed as follows:*

$$a = C(1+r)^n - p(1+r)^{n-1} - p(1+r)^{n-2} \text{ until } - p$$

The series of payments with their interests evidently form a geometrical series with the constant ratio $= (1+r)$ the payment $= p$, being the first term; we can therefore place its value at once instead of the series according to the expression found in section 108. The number of terms is evidently $= n$, because the payments are continued until the term p, which has not the common ratio in it. So we have for the amount of this series,

$$p \times \frac{(1+r)^n - 1}{(1+r) - 1}$$

and therefore for the value of the annuity,

$$a = C(1+r)^n - p \times \frac{(1+r)^n - 1}{(1+r) - 1}$$

$$= C(1+r)^n - p \times \frac{(1+r)^n - 1}{r}$$

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* To express this in a rule would be useless; we will rather substitute, by way of example, the numbers which the letters represent, and join the result in the first example following, taking the data of the foregoing example.

If the payments were to commence at a later period than the beginning, or to stop after a certain number of payments, as for instance, the supposed probability of the life of the person enjoying a life annuity, it is evident that the only difference resulting would be in the number of the years which denote the power of the ratio of the series of the payments. Suppose it should take place, *m* years after the lending of the money, or beginning of the compound interest upon the original capital; we would then have:

$$a = C(1+r)^n - p\,\frac{(1+r)^{n-m} - 1}{r}$$

This latter is usually called reversion.

1*st Example.* Supposing the capital which was given in the preceding section, and that an annual payment of $800 was to be made, beginning with the first year, and letting the number of years also be 6, we shall have the amount in the hands of the receiver of the money at the end of 6 years:

By the expression,

$$a = 7500\,(1{,}06)^6 - 800\,\frac{(1{,}06)^6 - 1}{0{,}06}$$

$$a = \$10639{,}9 - 5580{,}266 = \$5058{,}633$$

2*nd Example.* Suppose the same capital originally given, and the same payments, to begin 6 years after the placing of the money; what will be the amount after 14 years?

By substituting these numbers in their proper place we obtain:

$$a = 7500\,(1{,}06)^{14} - 800\,\frac{(1{,}06)^8 - 1}{0{,}06}$$

from which is obtained:

$$a = 16956{,}88 - 1275{,}08 = 15681{,}8$$

To find in this case the rate per cent. or the number of years, having given the other parts, will again require methods of calculation which lie out of the limits of this work; as may be judged from their form, and by reference to the preceding chapter, on geometrical series.

§ 112. The determination of the value of arrears of payments is calculated upon the same principle as the payments in the preceding case; because it is supposed that the money due at former times, and not paid, would have increased in the same manner; therefore the solution of these cases lies in the second part of the above, and the result is obtained by a mere change of denomination; thus:

The amount of all arrears due $= a$
The yearly payments $= p$
The rate per cent. interest $= r$
The number of years' arrears due $= n$

Gives the result of $a = p \dfrac{(1 + r)^n - 1}{r}$

Example. An annual payment of \$1000 being in arrear for 7 years, what is the amount to be paid, on the principle of compound interest, at the rate of 6 per cent. annually?

This gives $a = 1000 \dfrac{(1{,}06)^7 - 1}{0{,}06} = 1000 \,.\, \dfrac{0{,}5036303}{0{,}06}$

or $a = \$8393{,}33$

§ 113. When a certain capital is to be distributed into equal payments under the allowance of compound interest, as is often done, the expression of section 111 gives the principle of this distribution by the simple supposition that the second part of the expression, containing the amount of the yearly payments, with their compound interest, must be equal to the first, containing the capital with its compound interest. That is to say, we have

$$C (1 + r)^n = p \frac{(1 + r)^n - 1}{r}$$

This, considered as product of extremes and means in a geometric proportion, gives

$$C : p = (1 + r)^n - 1 : r (1 + r)^n$$

So we may determine with equal ease the yearly payment

= p, which will extinguish (or be equal to) a certain present amount $= C$, at the rate per cent. $= r$, in the number of years $= n$; and the present capital which such yearly payments will represent; for we have from this proportion:

$$p = C \frac{r\,(1+r)^n}{(1+r)^n - 1}; \quad \text{and} \quad C = p \frac{(1+r)^n - 1}{r\,(1+r)^n};$$

by the simple rule of three.

In substituting here, by way of example, the numbers found or given in section 111, the above expression would stand thus:

$$\text{Payment } \$800 = 5580{,}\ 266 \frac{0{,}06\ (1{,}06)^6}{(1{,}06)^6 - 1}$$

$$\text{Capital } \$5580{,}266 = 800 \frac{(1{,}06)^6 - 1}{0{,}06\ (1{,}06)^6}$$

The determination of the number of years that it will take to extinguish a debt by given yearly and equal payments, is another question that is beyond our present limits, for it is the same as that stated in section 109. This subject is therefore dismissed, and it is expected that any student, who has applied himself to this exposition of the principles of this kind of calculation, with the necessary understanding of the general principles of arithmetic taught in this book, will find no difficulty in solving any of the questions upon this subject coming under the head of the parts treated in this chapter.

CHAPTER IV.

Of Alligation, or Mixtures of objects of different Values.

§ 114. In retail mercantile concerns it often occurs, that it is desirable to ascertain the proportional value of a mixture of things of different values, which are given. Reflection upon what has been heretofore taught would point out the principle upon which such a proportional value may be determined. This value of the mixture being naturally a certain mean of all the component parts, this operation of arithmetic is usually called *Alligation Medial.*

The quantity of each component part multiplied by the price of its unit (what is usually called its value) evidently gives the influence of this part upon the general mixture. It might therefore be considered generally as acting exactly in the same way as the product of cause into time. The sum of all these products evidently constitutes the whole. Thus we might say in any number of things mixed,

$$C \times T + c \times t + ɔ \times ʇ + Ↄ \times ⊥ = E$$

the sum of all these uniting in the common effect $= E$. If, therefore, the mean effect, that is, the mean value of each individual thing, or unit, in the mixture is to be determined, this whole effect, that is, the sum of all the partial effects, is to be divided by the number of things mixed, or the objects acting in the general result; because by the mixing each part of the mixture it is intended to be brought to the same value, or intended to be considered as such. And the same will be the case in any union of effects of any kind, as labour and the time of its duration, or any such like.

This, expressed in the form of section 94, will give, considering C, (or the cause) as the objects, and (the time) T, as their value, the following general result:

$$\text{Mean} = \frac{C \times T + c \times t + ɔ \times ʇ + Ↄ \times ⊥}{C + c + ɔ + Ↄ}$$

Example. Suppose that a number of men work at a certain work during a month, as follows, viz: 6 men work 15 days each; 4 men work 19 days each; 12 men work 20 days each; and 10 men work 26 days each, during that time; on how many days' work, on an average, can one calculate for each man, in a month? This gives:

$$\text{Mean} = \frac{6 \times 15 + 4 \times 19 + 12 \times 20 + 10 \times 26}{6 + 4 + 12 + 10} = 20 + \frac{18}{16}$$

In this manner it may evidently also be calculated, that in a number of workmen engaged in a work the occasional absences may reduce the amount of work which they would otherwise perform; to the mere result of the product of the denominator of the above fraction into the quotient found, or the above workmen taken together, would in a month have executed only the work

$$W = 32\,(20 + \tfrac{13}{16}) = 666 \text{ days};$$

or the amount of the numerator of the fraction, as is evident; instead of which, if they had all been present the whole of the 26 working days in a month, they would have produced the work $= W = 26 \times 32 = 832$ days.

§ 115. When in such a composition it is desired to obtain a certain mean value of the objects mixed, or (as in the preceding example) a certain amount of work, by means of objects of different value, or, as above, men differently assiduous to their work, it becomes necessary to determine the quantity of each individual ingredient, (or, as above, the quantity of each men of a certain assiduity,) to obtain the desired aim, that is, the price of the thing aimed at, (or the number of days' work desired.) This operation of arithmetic is called *Alligation Alternate.* It is requisite that the quantity of objects below the mean value must compensate for those above it; their products must therefore become inverted, between every one above the mean in relation to every one below the mean, and inversely. In thus composing a mean without limitation of the quantity to be made up, or of any of the parts given, it is evident that a number of solutions will be possible for each question, but that all will be multiples of each other. The practical method used is the following:

The different values being written under each other, the difference between one value above the mean and this mean is taken, and placed opposite one of the values below the mean; and alternately, the difference between this lower value and the mean is written opposite to the value above the mean; thus all the differences that may be possible in the given case being taken, the numbers opposite to each value give the proportional compensation required of each ingredient above the mean, to compensate for each of those below and alternately; their sum must therefore in each case give the amount of compensation required of each on one side of the medium, referred to each on the other side, by which they are all reduced to a mean value; these numbers are therefore severally to be added, and give the quantity to be taken of each of these respective values, their products into the values, to which they are opposite, will give a sum answering a compound as desired. And every equal multiple of all the parts will also give an equal multiple of the whole. (The parts compared are linked, to show the operation.)

Example. A goldsmith having gold 15 carats fine, 19 carats, 21 carats, and 24 carats, wishes to make a mixture 20 carats fine; how much of each has he to take?

$$20 \left\{ \begin{array}{l} 15 \quad 4+1=5 \\ 19 \quad 1+4=5 \\ 21 \quad 1+5=6 \\ 24 \quad 5+1=6 \end{array} \right.$$

which gives

$$15\times5+19\times5+21\times6+24\times6=20\times22=440$$

or the whole mixture being 22, be it ounces, grains, or what it may, there must be in it 5 of the 15 carats gold; 5 of the 19; 6 of the 21; and 6 of the 24 carats gold; which evidently bears the proof of giving, when 20, the mean price, is multiplied by 22, the whole quantity mixed; the same result as is obtained by the sum of the individual products.

§ 116. If either the whole amount of the mixture, or any one of the parts to be mixed, is limited to a certain quantity, it becomes necessary, after the above operation, to take the ratio between the part given and its corresponding number in the above result, and to make all the other numbers in the like manner proportional to their corresponding ones in the above result.

1*st Example.* If in the above the whole mixture was required to be 36, instead of 22, we should have to make the proportions

$$22 : 36 = \left\{ \begin{array}{l} 5 : (\text{the 15 carats, or}) \ 8{,}45 \\ 5 : (\text{'' 19 ''}) \ 8{,}45 \\ 6 : (\text{'' 21 ''}) \ 9{,}818 \\ 6 : (\text{'' 24 ''}) \ 9{,}818 \end{array} \right.$$

2nd Example. A goldsmith has silver six ounces fine; 10 ounces fine; and 20 ounces of silver 9 ounces fine; how much of the two first must he add to the 20 ounces of 9 ounces fine, to make a mixture 8 ounces fine?

$$8\left\{\begin{array}{l}7\\9\\10\end{array}\right. \quad \begin{array}{ll}1+2&=3\\1&=1\\1&=1\end{array}$$

This will give the ratio of the silvers; now the silver at 9 ounces fine being determined at 20 ounces, the proportion formed from the ratio of the number found for that kind of silver, to the number limited for it, is that which must guide all the others, or gives the constant ratio, to which the ratio between the numbers just obtained and those to be employed must be made equal; as follows:

1 : 20 = 3 : (silver of 7 ounces fine =) 60
1 : 20 = 1 : (” 10 ” =) 20

§ 117. We shall now close these elements of arithmetic; for to go into more complicated practical applications would exceed the proper limits of first elements, and may be much better treated algebraically. The *regula falsi*, or rule of false supposition, both simple and compound, is intentionally omitted, the first because an attentive scholar of what has been here taught will not need it, but find in what he has learnt the better means to solve the question, the second because its operations belong more properly to algebra, so far as they actually lead to a determined result. A short retrospective view of what has been treated on in these elements may here not be misplaced.

§ 118. I have dwelt at some length upon the first elementary ideas of arithmetic, the notation or signs, of the arithmetic operations, and the principles of the systems of numeration, because, as was there said, these first elementary ideas, if well understood, will be of the greatest utility in rendering every operation in arithmetic easy; it is therefore to be wished, that the teacher extend them still more, by some practice upon other systems of numeration, besides the decimal system, and by familiarizing the varied combination of the signs of arithmetic, the full value of these combinations being ultimately assigned. The same reasons dictated to me the detailed description of the four rules of arithmetic, which it is certainly proper to make easy, and satisfactory to the mind of the beginner, if he is ever to know how to apply them in their proper place.

In treating vulgar fractions, I considered it obligatory upon me to proceed by exact mathematical demonstration, and to deduce them from their actual origin in an unexecuted division; while in decimal fractions the whole of their principles will at once spring from the consideration of division continued below the unit, according to the same system as above it. In considering all conventional subdivisions of the units of different kinds of quantities as denominate fractions, I found it possible to treat it with some system, which is not possible when each is treated separately. If I have deviated in these considerations from the usual method, I hope the clearness that results will excuse me. It appeared to me proper to bring the scholars to this point by what might be called theoretical steps.

§ 119. The Second Part will afford the scholar the satisfaction of a useful application of the principles learnt before. I considered it proper to devote a separate part of the book to this, in order to give the scholar the satisfaction of seeing how much he could do with the few elements he had learned, to present the connection between theory and practice in its proper light. Now this last is divested of its accessories to submit it to the principles of the theory, and thereby to present to the scholar an evident useful aim in his studies, that might encourage him to *mental* exertions.

§ 120. In the Third Part, treating of ratios and proportions, I considered myself both bound by true principle, and authorised by the progress of the scholar, to treat the subject as the beginning of the elements of the actual science of quantity; the principles being so few and simple, the task appeared to me, only to lay them well open to the scholar, and to show him all their bearings and consequences; a defective treatment of this part of arithmetic cannot but destroy, instead of cultivating, the reasoning and understanding of the scholar. These reasons determined me to a more detailed application to examples fully worked out, as they both help to explain the principles, and make their application pleasant.

The use of letters to denote a quantity before its determination, appeared to me proper to be introduced, so as gradually to habituate the scholar to more general considerations in regard to quantity, not servilely attached to the figures of our system of enumeration.

§ 121. After the steps made in the Third Part, I hope to need no excuse for the greater degree of generalization which has been introduced in the Fourth, except to say: that it was done with the avowed intention of leading the scholar imperceptibly into the entrance of algebra. It is absolutely useless to teach these parts by rules; no scholar ever remembers them; and he whose memory is mechanical enough for this, seldom knows where they are applicable. They are therefore useless to him; and to omit teaching properly the principles of these parts, is an injustice towards the student of arithmetic, who wishes to prepare himself by it for higher studies.

QUESTIONS FOR THE FOURTH PART.

ARITHMETIC PROGRESSION.

1. A sets out from one place, and B from another, 360 miles distant from A: they travel towards each other, so that A performs the first day 40 miles, the second 38, the third 36, and so on, decreasing his rate 2 miles daily. B begins the first day, and travels 20 miles, the second 22, increasing his rate 2 miles every day; in how many days will they meet? *Ans.* 6.

2. Find the sum of the series of natural numbers up to 100. *Ans.* = 5050.

3. Find the sum of the series of even numbers up to 100. *Ans.* = 2550.

4. Find the sum of the series of odd numbers up to 200. *Ans.* = 10,000.

5. Two travellers setting out together from the same place, A travels the first day 8 miles, and increases his rate four miles every day. B goes 25 miles per day from the beginning; how many days will they be in meeting again? *Ans.* 15,5 days.

6. It is required how far 125 stones must be placed at equal distance from each other, that the sum of their distance from a point 20 yards before the first, may be exactly equal to 5 miles?

7. How many eggs will be needed to lay 4 feet apart to occasion

a man who has to pick them up one by one, and bring them to a basket 3 yards behind the first, to have to walk 2,75 miles?

8. To find 7 arithmetic means between 6 and 46.

9. How many strokes does a clock strike in one whole day, by our common division of time, striking from 1 to 12. twice per day.

10. A man having to pick up 102 eggs laid in a row on the ground at one yard from each other, and carry them in a basket at two yards from the first, while another has to walk a distance of three miles from the same place and back again, which of the two has the advantage in the distance to be walked through? and how much?

Ans. The one who walks, has only 56 yards more to walk.

11. If one cent is placed on the first square of the chess board, and one more on each subsequent square, how many dollars will be on the whole board? *Ans.* $20,80.

GEOMETRIC PROGRESSION.

1. What is the sum of one hundred terms of the powers of 2?

2. What is the sum of 30 terms of the powers of 3?

3. The first term of a geometric series being 20, and the ratio ¼, what will the 25th term be?

ALLIGATION.

1. A man in a month of twenty-six working days works 6 days at the rate of $1,15 a day, 5 days at 75 cents per day, 3 days at $2 per day, 10 days at $1,50 a day, and is idle the 2 remaining days; at what rate per day does he earn, counting the whole of the 30 days in a month?

2. What is the fineness of a mixture of 2 oz. of gold 23 carats fine, 7 oz. 22 carats fine, 9 oz. 17 carats fine, and 3 oz. 20 carats fine?

3. A merchant sold a quantity of cloth, namely, 150 yards at $3,75, which cost him $3 per yard, 720 yards at $5 the yard, which cost him $3,75 per yard, 305 yards at $7,50 per yard, which cost him $6,35 per yard, and 100 yards at $2,50, which cost him $2 per yard; how much did he make per yard on an average?

4. A mixture is to be made of silver, some of which has cost $1,10 per oz., some 97 cents per oz., and the rest 88 cents per oz.; the mixture is to weigh 3 lbs.; how much is to be put in of each, to make the intrinsic value of the silver just a dollar an ounce?

COMPOUND INTEREST AND ANNUITY.

1. A man having an income of $5000 a year, saves one quarter of his income a year, which he puts to interest: what will be the amount of his savings in 12 years, at compound interest, at 5 per cent. per annum?

2. A trader living at the yearly expense of $500, and trading with the rest of his stock, augments it one third every year; at the end of the third year his stock is doubled; what was his original stock?

3. If a man spends every year his whole annual income, and one quarter of that sum in addition, which he takes from his capital bearing interest at 5 per cent.; how many years will he be in spending the whole capital itself, from the first beginning?

PROMISCUOUS QUESTIONS.

1. The sides of two square pieces of ground, are as 3 to 5, and the sum of their superficial content is 30600 square feet; what is the length of the sides of each? *Ans.* 90 feet and 150 feet.

2. Three young men entering into partnership, agree to make a common stock, to which each shall contribute in the ratio of the sum of the ages of the two other partners. A is 24 years old; B, 27 years; and C, 31 years; what will be the share of each?

3. A parcel of tobacco is sold, some at 12 cents per pound, the rest at 15 cents per pound; the proportion of the first to the latter, was as ¾ to ⅔, and the amount of the sale $380; how many pounds were there of each kind?

Ans. { Of the first, 1500 lbs.
Of the 2d, 1333⅓ lbs.

4. A grocer bought coffee, 3 bags of 80 lbs. at 21 cents per lb., 6 bags of 58 lbs. at 24 cents per lb., and 9 bags of 90 lbs. each, at 18 cents per lb., and sold the whole together at 22 cents the pound; what did he make by it? *Ans.* $47,84.

5. What fraction is that, to the numerator of which, if 1 be added it becomes ⅓, and if 1 be added to the denominator, it becomes ¼?

Ans. $\frac{4}{15}$.

6. The captain of a vessel, of which he owned ⅓, sold out the half of his share; he had before the sale $350 annual profit from it, besides his wages; how much remains to him annually after the sale of this part of his share?

7. A draper sold from a piece of cloth, ¼ at $5 the yard, one fifth at $4 per yard, and one sixth at $4,50 per yard; by this he obtained $168; how many yards were there in the piece? *Ans.* 60.

8. On the first of January 1793, a royalist in Europe agreed with a democrat to pay him 3 cents per day, until the restoration of the Bourbons, on condition of his paying him one louis d'or on $4,44, every day after that restoration. Taking the first of August, 1814, for the day of the return of the Bourbons, how would their account stand on the first of January, 1827, omitting all interest; and how, on calculating compound interest for every day from the epoch of these payments to the first of January, 1827, at 5 per cent. annually?

Without interest, the first would have paid $236,43.
the second $21760,44.

9. A merchant gains in trade such a sum, that $320 has the same ratio to it, as five times the sum has to $2500; what did he gain?

Ans. $400.

10. Two brothers comparing their ages, find that the sum of both ages is to that of the elder, as 19 to 7, and to 30, as 9 to the age of the other; what are their ages?

Ans. { 21
9.

11. The difference of the sides of two square rooms is to the side of the greater, as 2 to 6, and the difference of their square content, is = 128 feet; what are the sides of each of these rooms?

Ans. { 18.
14.

12. The profits of two men in their work, are as 8 to 5, and the

product of the numbers expressing their profits is 360; what was the profit of each? *Ans.* { $24. $15.

13. A joint stock is made up by five partners, who joining at different times, lay together as follows:

A has given	$15620,	which laid in the common stock	6 ys.	6 ms.
B	— 73921,	- - - - - -	4	2
C	— 39567,	- - - - - -	7	8
D	— 50220,	- - - - - -	8	4
E	— 6943,	- - - - - -	8	4

The capital is to be divided at the end of the time, and found to be double the amount put in by the stockholders; what is the share of each, in the whole amount?

14. Four merchants make a joint stock, under the following arrangements. A put in $15000, which remains 8 years and 6 months, during which time the association lasts; it is agreed, that as he is to take the chief direction, he shall have two per cent. previous of all profits, besides his share in the remaining profits. B puts in $20600 at the beginning. C puts in $13200 one year after the beginning; and as he is to work in the partnership, he is to have one and a half times the share of the profits which his capital would entitle him to, if he was not to work. D joins two and a half years after the beginning, with a capital of $60450; the partnership being dissolved, and the whole profits made being $80560, what is the share of each?

15. What will be the degree of heat of a mixture of 10 lbs. of water 100 degrees warm, 16 lbs. at 80 degrees warm, and 5 lb. at 212 degrees warm?

16. How much water 40 degrees warm must be added to the preceding mixture to make it 65 degrees warm?

17. If one man travels 52 miles in a day, walking 12 hours, and another 61 miles in 11 hours, what will be gained in time, on each, by sending both at the same time to meet from two places 180 miles from each other, to exchange dispatches, instead of sending them each the whole distance?

18. If 21000 bushels of oats last 200 horses at ½ bushel a day, 21 days, how long will 3700 bushels last to 760 horses at ⅝ of a bushel per day?

19. What provision must be made for an army of 9560 men in bread, if they shall receive 2 lb. per day for 70 days; if experience shews that 5000 men will in 25 days use 312500 lb. at 2½ lb. per day?

20. The product of two numbers is 63, and the square of their sum is to the square of their difference as 64 to 1, what are the numbers?
Ans. 9 & 7.

21. The difference of two numbers when divided by the lesser is equal to 48 divided by the greater; and when divided by the greater is equal to 3 divided by the lesser; what are the two numbers?
Ans. 4 & 16.

TABLES

Of the Proportional Subdivisions, or Denominate Fractions, of Weights, Measures, Time, &c.

Explanation.—In the following tables, the denominations of the subdivisions will be found written in full, at the head of each table, and in their usual abreviations within the tables themselves. The first number of each square, is the number of units of each subdivision required to make the unit of the kind found at the right hand; and the lower number in the same square, is the decimal fraction corresponding to the same subdivision and unit, carried to 7 decimals.

TIME.

Seconds.	*Minutes.*	*Hours.*	*Days.*	*Years.*
″ 60 0, 016666	′ 1			
3600 0, 0002777	60 0, 016666	*h.* 1		
86400 0, 0011574	1440 0, 0069444	24 0, 041666	*d.* 1	
	5259487, 8	8765, 813	*d. h. m. s.* 365. 5. 48. 48	*y.* 1

CIRCULAR PARTS.

Seconds.	*Minutes.*	*Degrees.*	*Circumference.*
″ 60 0, 016666	′ 1		
360 0, 0002777	60 0, 016666	° 1	
1296000 0, 00000077	21600 0, 0000462	360 0, 002777	*c.* 1

LONG MEASURE.

Inches.	*Feet.*	*Yards.*	*Fathoms.*	*Poles.*	*Furlongs.*	*Miles.*	*Leagues.*	*Degree of the Equator.*
in. 12 0, 08333	*ft.* 1							
36 0, 0277777	3 0, 333333	*yd.* 1						
72 0, 0138888	6 0, 1666666	2 0, 5	*fth.* 1					
198 0, 0050505	16, 5 0, 0606061	5, 5 0, 1818182	2, 75 0, 3636364	*p.* 1				
7920 0, 0001263	660 0, 0015151	220 0, 0045454	110 0, 0990909	40 0, 025	*f.* 1			
63360 0, 0000158	5280 0, 0001893	1760 0, 0005680	880 0, 0011361	320 0, 0031350	8 0, 125	*m.* 1		
190080 0, 0000053	15840 0, 0000631	5280 0, 0001893	2640 0, 0003787	960 0, 0010417	24 0, 0416666	3 0, 3333333	*l.* 1	
4384512 0, 0000023	365376 0, 0000274	121792 0, 0000821	60896 0, 0001642	22144 0, 0004516	553, 6 0, 0018064	69, 2 0, 0144509	23, 066 0, 0433526	° 1

SQUARE MEASURE.

Inches.	*Feet.*	*Yards.*	*Poles.*	*Roods.*	*Acres.*	*Miles.*	*Leagues.*
in. 144 0, 0069165	*ft.* 1						
1296 0, 0007777	9 0, 111111	*yd.* 1					
39204 0, 0000257	272, 25 0, 0036739	30, 25 0, 0330578	*p.* 1				
1568160 0, 0000006	10890 0, 0000918	1210 0, 0008264	40 0, 025	*r.* 1			
6272640 0, 0000002	43560 0, 0000229	4840 0, 0002066	160 0, 00625	4 0, 25	*a.* 1		
	27794400 , 00000003	3089600 0, 0000003	102400 0, 0000977	2560 0, 0003906	640 0, 0015625	*m.* 1	
		27878400 , 00000003	921600 0, 0000011	23040 0, 0000434	5760 0, 0001736	9 0, 1111111	*l.* 1

CUBIC MEASURE.

Inches.	*Feet.*	*Yards.*	*Fathoms.*
in. 1728 0,0005787	*f.* 1		
46656 0,00002143	27 0,037037	*y.* 1	
376648 0,00000265	216 0,0046296	8 0,125	*fth.* 1

CLOTH MEASURE.

Inches.	*Nails.*	*Quarters.*	*Yards.*	*Ells.*
in. 2,25 0,4444	*nl.* 1			
9 0,111111	4 0,25	*qr.* 1		
36 0,02777	16 0,0625	4 0,25	*y.* 1	
45 0,02222	20 0,05	5 0,2	1,25 0,8	*e.* 1

DRY MEASURE.

Pints.	*Gallons.*	*Pecks.*	*Bushels.*
pl. 8 0,125	*g.* 1		
16 0,0625	2 0,5	*pk.* 1	
64 0,015625	8 0,125	4 0,25	*b.* 1

Eight Bushels make a Quarter; but as this is not used in any part of this country, any more than the Wey and Last, we have omitted them.

WINE MEASURE.

Pints.	*Quarts.*	*Gallons.*	*Tierces.*	*Hogsheads.*	*Puncheons.*	*Pipes.*	*Tuns.*
p. 2 0,5	*q.* 1						
8 0,125	4 0,25	*g.* 1					
336 0,0029762	168 0,0059524	42 0,0238095	*ti.* 1				
504 0,0019841	252 0,0039682	63 0,015873	1,5 0,66666	*hhd.* 1			
672 0,0014881	336 0,0029762	84 0,0119047	2 0,5	1,5 0,66666	*pun.* 1		
1008 0,0009920	504 0,0019841	126 0,0079365	3 0,333333	2 0,5	1,5 0,666666	*pip.* 1	
2016 0,0004960	1008 0,0009920	252 0,0039682	6 0,166666	4 0,25	3 0,33333	2 0,5	*tun.* 1

Habit alone determines, in different countries where these measures are used, to which purposes the two different measures of liquids are applied besides the two liquids of which they bear the name, and these habits vary from time to time. In the state of New-York, Beer measure is little used, but the ordinary measure for all liquids is Wine measure.

BEER MEASURE.

Pints.	*Quarts.*	*Gallons.*	*Barrels.*	*Hogsheads.*	*Butts.*
p. 2 0, 5	*q.* 1				
8 0, 125	4 0, 25	*g.* 1			
288 0, 00347	144 0, 00694	36 0, 02777	*b.* 1		
482	216	54	1, 5	*hhd.* 1	
864	432	108	3	2	*b.* 1

TROY WEIGHT.

(Used for Gold, Silver, Jewels, and retail dealing.)

Grains.	*Pennyweights.*	*Ounces.*	*Pounds.*
gr. 24 0, 0416666	*dwt.* 1		
480 0, 0020833	20 0, 05	*oz.* 1	
5760 0, 0001736	240 0,0041666	12 0, 08333	*lb.* 1

APOTHECARIES' WEIGHT.

(Used in compounding Medicines.)

Grains.	*Scruples.*	*Drachms.*	*Ounces.*	*Pounds.*
gr. 20 0, 05	*sc.* 1			
60 0, 016666	3 0, 333	*dr.* 1		
480 0, 0020833	24 0, 041666	8 0, 125	*oz.* 1	
5760 0, 0001736	288 0, 0034722	96 0, 0104165	12 0, 08333	*lb.* 1

AVOIRDUPOIS WEIGHT.

Drachms.	*Ounces.*	*Pounds.*	*Quarters.*	*Cwt.*	*Tons.*
dr.					
16	*oz.*				
0,0625	1				
256	16	*lb.*			
0,0039014	0,05555	1			
7168	448	28	*qr.*		
0,0001395	0,0022321	0,0357143	1		
28672	1792	112	4	*cwt.*	
	0,0005580	0,0090178	0,25	1	
573440	35840	2240	80	20	*ton.*
		0,0004464	0,0125	0,05	1

This kind of weight is used in every other case of mercantile transaction, whether in the great transactions of general commerce, or in the retail trade.

	oz.	*dwt.*	*gr.*	
1 *lb.* Avoirdupois =	14.	11.	16	Troy.
1 *oz.* „ =		18.	5½	„
1 *dr.* „ =		1.	3⅓	„

Before the last law in England, of 1825, regulating weights and measures, the following were the cubic contents of the different measures of capacity; viz.:

The Bushel, 2150⅖ cubic inches = a cylinder 8 in. deep, 18,5 in. diameter.

The Gallon, dry measure, 268⅘ cubic inches.

„ „ for beer, 282 „ „

„ „ for wine, 231 „ „

These two latter gallons have to each other the same ratio as the weights Avoirdupois and Troy.

By the law of 1825, the Bushel contains 2217,6 cubic inches.
the Gallon „ 277,2 „
and is used indiscriminately for dry and liquid measure.

The capacities are determined, not by measurement of the cubic contents, but by the weight of pure water at the temperature of 62° of Fahrnheit's thermometer contained in the vessels; the bushel holding 80, and the gallon 10 lbs. avoirdupois.

The standard of lineal measure of the State of New-York is the yard. Its length is by statute determined from the pendulum vibrating seconds at Columbia College, to which it bears the ratio of one million to one million eighty-six thousand one hundred and forty-one = 1 : 1,086141.

The pendulum being found by the experiments of Sabine to be 39,10107 inches.

The unit of weight is the pound, of such magnitude that a cubic foot of water at its maximum density, weighs sixty-two and a half pounds, or one thousand ounces.

The unit of measures of capacity is the gallon, which holds exactly ten pounds of water, at the maximum of density; the bushel holds exactly eighty pounds of water at its maximum density.

This statute was passed in September, 1827, and will go into effect on the 1st January, 1829.

The places with which the most frequent transactions of Exchange are performed in the United States, and the relative value of their money as employed by the Custom-house in determining the prices of goods, are as follows:

Names of Places.	*Values in Dollars.*	*Values of the Coins.*	*Denominations of Values in the Countries.*			
			Weights.		Long Measures.	
			U. States.	*Foreign.*	*U. States.*	*Foreign.*
England,	4,44	1 Pound sterling (£)				
	1,	4*s*. 6*d*. sterling,				
Ireland,	4,1	1 Pound Irish. (This is abolished by law in Ireland.)				
France,	0,18173125	1 Livre tournois.				
	0,18333	1 Franc.				
Netherlands,	0,40	1 Guilder........	....	....	75 Yards.	100 Ells.
Hamburg,	0,33333	1 Mark banco........	109,5 lb.	100 lb.		
Denmark,	1,	1 Rix dollar........				
Spain,	0,10	1 Real plate	25 lb.	1 Arobe.		
——	0,05	1 —— vellon	97 lb.	100 lb.		
Portugal,	1,24	1 Milrea........	....	....	33,125 in 26⅔ in.	1 Vara. 1 Canade.
China,	1,48	1 Tale.				
Bengal,	0,58	1 Rupee.				
Bremen,	0,75	1 Rix dollar........	110 lb.	100 lb.		
Antwerp,	0,40	1 Guilder	100 lb.	90 lb.	74 Yards.	100 Ells.
	0,01	1 Groat				
Russia,	0,55	1 Ruble........	88,75 lb.	100 lb.	7 Yards.	9 Archins

Value of Foreign Coins according to the Laws of the United States.

Gold coins of G. Britain and Portugal are rated at $1 for 27 grains.
" France " " 27½ "
" Spain " " 28½ "

Course of France.	*gives or rec's*	*Quantity.*	*Denomination.*	*Quantity.*	*Denomination.*
Amsterdam,	r	3	Francs, ...	56	Deniers groats.
Antwerp,	r	3	Francs, ...	55, 25	Deniers groats.
ditto	g	100	Francs, ...	99, 50	Francs.
Augsburg,	r	300	Francs, ...	117	Florins couran
ditto	g	1	Florin Ct, ...	255	Centimes.
Basil and Zuric,	r *or* g	100	Francs, ...	99, 50	Francs.
Berlin,	r	300	Francs, ...	78	Rix dollars.
ditto	g	£ 1	Banco Prussian, ..	4, 80	Francs.
Bologna,	r	3	Francs, ...	56	Sols.
Constantinople	r	300	Francs, ...	102, 5	Piastres.
ditto	g	1	Piastre, ...	2, 90	Francs.
Copenhagen,	r	300	Francs, ...	65, 5	Centimes.
ditto	g	1	Rix Dollar, ...	4, 45	Francs.
Frankfort on Main,	r	300	Francs, ...	79	Rix dollars.
		100	Francs, ...	99, 5	Francs.
ditto	g	100	Francs, ...	99, 5	Francs.
Genoa,	g	1	Piastre (of 115s. h. b.)	4, 80	Francs.
Geneva,	g	100	Livres courant, ...	166	Francs.
Hamburg & Altona,	g	100	Mark banco......	190	Francs.
	r	3	Francs, ...	24	Sols Lubs.
Leipzig & Dresden	r	300	Francs, ...	76	Rix dollars.
Lisbon,	r	3	Francs, ...	480	Reas.
Leghorn,	g	1	Piastre (of 8 Reals)	5, 10	Francs.
London,	g	1	£ sterling, ...	23	Francs.
	r	3	Francs, ...	21	Pence sterling.
Milan,	r	3	Francs, ...	55	Sols imperial.
		6	Francs, ...	7 lire 15	Sols courant.
	g	1	Lire imperiale, ...	1, 09	Francs.
Naples,	g	1	Ducat (of 10 Carlins,)...	4, 20	Francs.
New-York,	r	1	Franc, ...	0, 183	Dollars.
Palermo,	r	1	Franc, ...	46	Grains.
Petersburg,	g	1	Ruble, ...	4, 40	Francs.
Spain,	g	1	Pistole (of 32 reals)	15	Francs.
		1	Piastre (of 8 reals)	3, 75	Francs.
Stockholm,	r	3	Francs, ...	25	Shillings.
Turin,	r	3	Francs, ...	50	Sols Piedmont.
Venice,	r	300	Francs, ...	61	Ducats banco
Vienna,	r	1	Franc, ...	23	Cruezers.
	g	1	Florin, ...	2, 55	Francs.

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